WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2022

MTMACOR14T-MATHEMATICS (CC14)

RING THEORY AND LINEAR ALGEBRA II

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5=10$

(a) Let $f(x)=x^4+x^3-3x^2-x+2$ and $g(x)=x^4+x^3-x^2+x-2$. Find the gcd of $f(x)$ and $g(x)$, as polynomials over $\mathbb{Q}$.

(b) Let $f(x)=x^6+x^3+1\in\mathbb{Z}[x]$. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

(c) Is $\mathbb{Z}[\sqrt{-5}]$ a UFD? Justify.

(d) Let $\beta=\{(2,1),(3,1)\}$ be an ordered basis for $\mathbb{R}^2$. Suppose that the dual basis of β is $\beta^*=\{f_1, f_2\}$. Find $f_1(x,y)$ and $f_2(x,y)$.

(e) Consider the matrix $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ in $M_{2\times2}(\mathbb{R})$. Is the given matrix diagonalizable? Justify.

(f) In an inner product space V, show that $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$, $\forall\, x, y\in V$.

(g) Let V be an inner product space and let T be a normal operator on V. Show that $\|T(x)\|=\|T^*(x)\|$, $\forall\, x\in V$.

(h) Show that any orthonormal set of vectors in an inner product space V is linearly independent.

2. (a) Let R be a UFD and $f(x), g(x)$ be two primitive polynomials in $R[x]$, then prove that $f(x)\,g(x)$ is also a primitive polynomial. 4

(b) Let R be the ring $\mathbb{Z}\times\mathbb{Z}$. Solve the polynomial equation $(1,1)x^2-(5,14)x+(6,33)=(0,0)$ over R. Show that the linear equation $(5,0)x+(20,0)=(0,0)$ has infinitely many roots in R. 2+2

3. (a) Let R be a ring with unity. Show that $R[x]/\langle x\rangle\simeq R$. 4

(b) Let R be a principal ideal domain and $p\in R$. Show that p is irreducible if and only if p is prime. 4

4. (a) Let $f(x) \in F[x]$ be a polynomial of degree 2 or 3, where F is a field. Show that $f(x)$ is irreducible over F if and only if $f(x)$ has no zero in F. 4

(b) Show that $f(x) = x^4 - 2x^3 + x + 1$ is irreducible in $\mathbb{Q}[x]$. 4

5. (a) Let V be an n-dimensional inner product space and W be a subspace of V. Then prove that $\dim(V) = \dim(W) \oplus \dim(W^{\perp})$, where $W^{\perp}$ denotes the orthogonal complement of W. 5

(b) Define $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ by $T(p(x) = (p(0), p(2))$, where $P_1(\mathbb{R})$ is the polynomials of degree atmost 1 over $\mathbb{R}$. Let β and γ be the standard ordered bases for $P_1(\mathbb{R})$ and $\mathbb{R}^2$ respectively. Find $[T]_{\beta}^{\gamma}$ and $[T^t]_{\gamma^*}^{\beta^*}$. Also show that $[T^t]_{\gamma^*}^{\beta^*} = ([T]_{\beta}^{\gamma})^t$. 3

6. Let T be the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = f(x) + (x+1)f'(x)$ and let β be the standard ordered basis for $P_2(\mathbb{R})$ and let $A = [T]_{\beta}$. Find the eigen values and the eigen vectors of T. Examine whether T is diagonizable or not. 2+4+2

7. (a) Let T be a linear operator on $\mathbb{R}^4$ defined by $T(a, b, c, d) = (a + b + 2c - d, b + d, 2c - d, c + d)$ and let $W = \{(t, s, 0, 0) : t, s \in \mathbb{R}\}$. Show that W is a T-invariant subspace of $\mathbb{R}^4$. 3

(b) Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be distinct eigen values of T. If $v_1, v_2, \cdots, v_k$ are eigen vectors of T such that λ_i corresponds to $v_i (1 \le i \le k)$, then show that $\{v_1, v_2, \cdots, v_k)$ is linearly independent. 5

8. (a) Let T be a linear operator on a finite dimensional vector space V and let $f(t)$ be the characteristic polynomial of T. Then prove that $f(T) = T_0$, where T_0 denotes the zero transformation. 4

(b) Let $\langle \, , \rangle$ be the standard inner product on $\mathbb{C}^2$. Prove that there is no nonzero linear operator on $\mathbb{C}^2$ such that $\langle \alpha, T\alpha \rangle = 0$ for every α in $\mathbb{C}^2$. Generalize this result for $\mathbb{C}^n$, where n is any positive integer greater equal to 2. 4

9. (a) Apply Gram-Schmidt process to the subset $S = \{(2, -1, -2, 4), (-2, 1, -5, 5), (-1, 3, 7, 11)\}$ of the inner product space $\mathbb{R}^4$ to obtain an orthogonal basis for span(S). Then normalize the vectors in this basis to obtain an orthonormal basis β for span(S). 5

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator whose matrix representation in the standard ordered basis is given by 3

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \; 0 < \theta < \pi .$$

Show that T is normal.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

——×——