WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2022

MTMACOR13T-MATHEMATICS (CC13)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* questions from the rest

1. Answer any ***five*** questions from the following: 2×5 =10

(a) Let (X, d) be a metric space and $x_1, x_2, \cdots, x_n \in X$. Prove that $d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$.

(b) Let (X, d) be a metric space. Prove that $\{x\}$ is a closed subset of X for all $x \in X$.

(c) Let $C[0,1]$ be the set of all continuous real valued functions on the closed interval $[0,1]$. Assume that d_1 and d_∞ are two metrics on $C[0,1]$ where for all

$$f, g \in C[0,1], \quad d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx, \quad d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

Compute $d_1(f, g)$ and $d_\infty(f, g)$ for the functions

$$f(x) = x, \; g(x) = x^2 \text{ for all } x \in C[0,1].$$

(d) Show that the metric defined by $d(x, y) = |\tan^{-1} x - \tan^{-1} y|$ on $\mathbb{R}$ is a bounded metric.

(e) Show that $f(z) = \bar{z}$, $\forall\, z \in \mathbb{C}$ is a continuous function.

(f) At which point the function $f(z) = |z|^2 + i\bar{z} + 1$ is differentiable?

(g) If an analytic function $f(z)$ is such that $\text{Re}\{f'(z)\} = 2y$ and $f(1+i) = 2$ then find the imaginary part of $f(z)$.

(h) Find the region of convergence of the series $\sum_{n=1}^{\infty} n! z^n$.

2. Let X be the set of all real sequences. Define $d : X \times X \to \mathbb{R}$ by 1+5+2

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \times \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

for all $x = \{x_n\}_n$, $y = \{y_n\}_n \in X$. Show that d is well defined. Prove that d is a metric on X. Is (X, d) a bounded metric space? Justify your answer.

3. (a) Let (X, d) be a metric space with $A \subset X$. Show that if $x \notin A$ and x is a limit point A then $d(x, A) = 0$. 4

(b) Define Cauchy sequence in a metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space with $d(x_n, y_n) \to 0$ as $n \to \infty$. If $\{x_n\}$ is a Cauchy sequence in X, prove that $\{y_n\}$ is also a Cauchy sequence. 4

4. (a) Prove that the space $C[a, b]$ of all continuous real-valued functions defined on $[a, b]$ with the metric $d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$ is a complete metric space. 4

(b) Prove that a closed subset of a compact metric space (X, d) is compact. 4

5. (a) Let $f, g : (X, d_X) \to (Y, d_Y)$ be continuous functions. Prove that $\{x \in X : f(x) = g(x)\}$ is a closed subset of X. 4

(b) Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous function where X is connected. Prove that $f(X)$ is a connected subset of Y. 4

6. (a) Show that the function 4

$$f(z) = \begin{cases} \dfrac{|z|}{\operatorname{Re} z}; & \operatorname{Re} z \neq 0 \\ 0 \ ; & \operatorname{Re} z = 0 \end{cases}$$

is not continuous at $z = 0$.

(b) Let $\alpha \in \mathbb{C}$. Show that the function $f(z) = |z - \omega|^2$, $\forall z \in \mathbb{C}$ is differentiable only at α. 4

7. (a) If $f(z) = \begin{cases} e^{-z^{-4}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ show that $f(z)$ is not analytic at $z = 0$ although C.R. equations are satisfied at the point $z = 0$. 4

(b) Let $f(z) = u(x, y) + i\, v(x, y)$ be differentiable at a point $z_0 = x_0 + y_0$. Then prove that the first order partial derivatives $u_x(x_0, y_0)$, $u_y(x_0, y_0)$, $v_x(x_0, y_0)$, $v_y(x_0, y_0)$ exist and they satisfy Cauchy-Riemann equations at a point (x_0, y_0). Find $f'(z)$. 4

8. (a) Evaluate $\int_C \dfrac{e^z}{(z+2)(z+1)^2} dz$ where C is the circle $|z| = 3$. 4

(b) Let f be an analytic function in a simply connected region D in the complex plane and let α, β be any two points in D. Prove that $\int_\alpha^\beta f(z) dz$ is independent of the path joining α and β. 4

9. (a) Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ where C is the circle $|z| = 3$ 4

(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2}$ is convergent in the region $1 < |z| < 2$. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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