WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 2nd Semester Examination, 2022

MTMACOR03T-MATHEMATICS (CC3)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Using Archimedean property of $\mathbb{R}$, prove that the set of natural numbers, $\mathbb{N}$ is unbounded above.

(b) Find the supremum of the set $S=\left\{\frac{1}{p}+\frac{1}{q}\ :\ p,\ q \in \mathbb{N}\right\}$.

(c) For any two sets S and T in $\mathbb{R}$, prove that $\overline{S\cap T} \subseteq \overline{S}\cap\overline{T}$, where for any $A\subseteq\mathbb{R}$, $\overline{A}$ denotes the closure of A.

(d) If $A=\left[\frac{1}{3}, \frac{8}{3}\right]$ and $B=\left(1, \frac{11}{3}\right)$, examine whether $A\cup B$ is compact or not.

(e) Find the set of all limit points of the set $E=\left\{\frac{n-1}{n+1}\ :\ n\in\mathbb{N}\right\}\cup\{2,\ 3\}$.

(f) Two sets A and B of real numbers are such that A is closed and B is compact. Prove that $A\cap B$ is compact.

(g) Show that $\left(\frac{n}{n+1}\right)_n$ is a Cauchy sequence.

(h) Apply Cauchy's root test to check the convergence of the series:

$$1+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^4}+\ldots\ldots..$$

2. (a) Let T be a bounded subset of R. If $S=\{|x-y|:\ x,\ y\in T\}$ then show that $\sup S=\sup T-\inf T$. 3

(b) Prove that the set of rational numbers is not order complete. 3

(c) If A be an uncountable set and B be a countable subset of A, then prove that $A-B$ is uncountable. 2

3. (a) Give example of a set which is — 2+2
   (i) both open and closed,
   (ii) neither open nor closed.
   Give reasons in support of your answer.
   (b) Prove that every open interval is an open set and every open set is an union of open intervals. — 2+2

4. (a) Let $H=(0,1)$ and $x\in H$. Let $\sigma=\{I_x : x\in H\}$, where $I_x=\left(\frac{x}{2},\frac{x+1}{2}\right)$. Show that σ is an open cover of H but it has no finite sub cover. — 3
   (b) If S and T are compact sets in R then show that $S\cup T$ is also compact. — 2
   (c) Prove or disprove: Union of an infinite number of compact sets is compact. Give reasons in support of your answer. — 3

5. (a) State Bolzano-Weierstrass theorem for the set of real numbers. Can you apply the theorem for the set of natural numbers? Justify your answer. — 1+1
   (b) Show that the intersection of finite collection of open sets is an open set in $\mathbb{R}$. Give an example to show that arbitrary intersection of open sets may not be an open set. — 2+1
   (c) Show that the set $A=\{x\in\mathbb{R} : \cos x\neq 0\}$ is an open set, but not a closed set. — 2+1

6. (a) Show that the sequence $\left\{\frac{3^{2n}}{4^{3n}}\right\}$ is a null sequence. — 2
   (b) Use Sandwich theorem to prove the following limit: — 3

$$\lim_{n\to\infty}\left[\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots\cdots+\frac{n}{n^2+n}\right]$$

   (c) If $x_1=8$ and $x_{n+1}=\frac{1}{2}x_n+2$ for all $n\in N$, then show that the sequence $\{x_n\}$ is monotonically decreasing and bounded. Find limit. — 3

7. (a) Use Cauchy's criterion of convergence to examine the convergence of the sequence $\{x_n\}$ where — 2

$$x_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots\cdots+\frac{1}{n!}$$

   (b) If the n-th term of the sequence $\{x_n\}$ is given by $x_n=\frac{n}{2}-\left[\frac{n}{2}\right]$, where $[x]$ is the greatest integer not greater than x, then find two subsequences of $\{x_n\}$, one of which converges to the upper limit and the other converges to the lower limit of $\{x_n\}$. — 2
   (c) Show that every Cauchy sequence is bounded. Is the converse true? Give reasons in support of your answer. — 2+2

8. (a) If $a_n > 0$ for all $n \in \mathbb{N}$ and if the sequence $(n^2 a_n)_n$ is convergent, show that the infinite series $\sum a_n$ is convergent. 2

(b) For any positive number α, apply Cauchy's root test to check the convergence of the series $\sum a_n$ where for all $n \in \mathbb{N}$, 4

$$a_n = \left(1 + \frac{1}{n^\alpha}\right)^{-n^{\alpha+1}}$$

(c) Use the ratio test to check the convergence of the series 2

$$1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \ldots\ldots\ldots$$

9. (a) Let $(u_n)_n$ be a sequence of positive terms such that the infinite series $\sum u_n$ is convergent. Use comparison test to show that $\sum u_n^2$ is also a convergent series. 2

(b) Define absolute convergence of an infinite series of real numbers. Show that every absolutely convergent series is convergent. 1+2

(c) Use Leibnitz test to show that the alternating series $\sum (-1)^n \left[\sqrt{n^2+1} - n\right]$ is convergent. Show by comparison test (limit form) that this alternating series is not absolutely convergent. 1+2

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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