WEST BENGAL STATE UNIVERSITY

B.Sc. Honours Part-II Examination, 2022

MATHEMATICS

PAPER: MTMA-III

Time Allotted: 4 Hours — Full Marks: 100

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

GROUP-A

Answer any *three* questions from the following — 5×3 = 15

1. Define special roots of $x^n - 1 = 0$. Prove that the special roots of the equation $x^n - 1 = 0$ are the roots of a reciprocal equation. — 2+3

2. Solve by using Cardan's method: — 5

$$x^3 + 3x^2 - 3 = 0$$

3. Solve by Ferraris method: — 5

$$x^4 + 3x^3 + x^2 - 2 = 0$$

4. (a) If x, y, z are positive rational numbers then show that — 3

$$\left(\frac{x^2 + y^2 + z^2}{x + y + z}\right)^{x+y+z} \geq x^x y^y z^z$$

(b) If a, b, c are three positive numbers in harmonic progression and n is a positive integer greater than 1, prove that — 2

$$a^n + c^n > 2b^n$$

5. Reduce the reciprocal equation $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ to the standard form and solve it. — 5

6. (a) Find greatest value of $(2x+1)^3(y+2)^2$ when $x + y = 3$ and $-\frac{1}{2} < x < 5$. — 3

(b) If a, b, c are unequal positive numbers such that sum of any two numbers is greater than the third then show that — 2

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} > \frac{9}{a+b+c}$$

GROUP-B

Answer any *one* question from the following 10×1 = 10

7. (a) Show that if the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ multiplied three times itself, then it will give an identity permutation. 2

(b) If $a = (1\ \ 2\ \ 3\ \ 4)$ then show that the set $\{a, a^2, a^3, a^4\}$ forms a cyclic group. 3

(c) Show that every proper subgroup of symmetric group S_3 is cyclic. 3

(d) Find the images of the element 3 and 4 if $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & & & 1 & 4 \end{pmatrix}$ be an even permutation. 2

8. (a) If G be a group and H be a subgroup of G then prove that any two left cosets of H in G are either identical or they have no common element. 3

(b) Prove that every group of prime order is cyclic. 3

(c) Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$ as a product of transpositions. Find also order of the permutation. 2

(d) Show that if two right cosets Ha and Hb be distinct then two left cosets $a^{-1}H$ and $b^{-1}H$ are distinct. 2

GROUP-C

Answer any *two* questions from the following 10×2 = 20

9. (a) Let P stands for the set of all the functions from [0, 1] into $\mathbb{R}$, which are differentiable over [0, 1]. Show that P is a vector space over $\mathbb{R}$ if addition of functions in P and multiplication of functions in P by the elements of $\mathbb{R}$ are defined pointwise on [0, 1]. 3

(b) Find a basis and dimension of the subspace W of $\mathbb{R}^3$ where 2+1

$$W = \{(x, y, z) \in \mathbb{R}^3 :\ x + y + z = 0\}$$

(c) State Cayley-Hamilton theorem. Hence compute A^{-1} where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$. 1+3

10.(a) Correct or Justify: 3

Let V be a vector space over a field F. Let U and W be two subspaces of V such that $\dim U = \dim W$. Then $U = W$.

(b) Prove that in an Euclidean vector space, $\|\alpha + \beta\| \le \|\alpha\| + \|\beta\|$. 3

What happens when the equality holds?

(c) Use Gram-Schimdt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with standard inner product, generated by the linearly independent set $\{(1, 1, 0, 1),\ (1, 1, 0, 0),\ (0, 1, 0, 1)\}$. 4

11.(a) Investigate for what values of a and b the following system of equations 4

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

has (i) only one solution, (ii) no solution and (iii) an infinite number of solutions.

(b) Give an example to show that union of two vector subspaces of a vector space $V(F)$ may not be a subspace of $V(F)$. 2

(c) If in a vector space $V(F)$ of dimension n, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of generators, prove that $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is basis of V. 4

12.(a) Prove that every non-zero orthogonal set of vectors of a Euclidean space is linearly independent. 3

(b) Prove that the solutions of a homogeneous system $\mathbf{AX} = \underline{\mathbf{0}}$ in n unknowns where $\mathbf{A}$ is an $m\times n$ matrix over a field F, form a subspace of $V_n(F)$. 3

(c) Find an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P^{-1}AP}$ is a diagonal matrix, where 4

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

GROUP-D

Answer any *two* questions from the following 10×2 = 20

13.(a) Prove that every bounded sequence of real numbers has a convergent subsequence. Is the result true if the word ‘bounded’ be replaced by ‘bounded below’? — Justify. 3+1

(b) Define upper and lower limits of a bounded sequence. Determine these limits for the sequence $\{a_n\}_n$ where $a_n = (-1)^n + \dfrac{1}{n+1}$. 3

(c) Prove that if the subsequence $\{x_{3n}\}_n$ of a monotonic sequence $\{x_n\}$ converges to l then every subsequence of $\{x_n\}$ converges to l. 3

14.(a) Examine the convergence of the series 3

$$\sum_n \left\{ \frac{2\cdot 4\cdot 6\cdots\cdots 2n}{3\cdot 5\cdot 7\cdots\cdots (2n+1)} \right\}^2$$

(b) If $\{b_n\}$ is a monotone bounded sequence and $\sum_n a_n$ is convergent series then prove that $\sum a_n b_n$ is convergent. 4

(c) Examine the convergence of the series $\sum \frac{(-1)^n}{(n+1)\log(n+1)}$. 3

15.(a) Let $f:[a, b] \to \mathbb{R}$ be continuous on a closed interval $[a, b]$ and $f(a)\cdot f(b) < 0$. Prove that there exists at least one point c in the open interval (a, b) such that $f(c) = 0$. 4

(b) Find a and b in order that $\lim\limits_{x\to 0} \frac{a\sin(2x) - b\sin(3x)}{5x^3} = 1$. 3

(c) Prove that $f(3)$ is a minimum value of 3

$$f(x) = |3-x| + |2+x| + |5-x|, \quad x \in \mathbb{R}$$

but $f'(3)$ does not exist.

16.(a) State and prove Cauchy's Mean Value theorem and deduce from it Lagrange's Mean Value theorem. 1+3+1

(b) $f: \mathbb{R} \to \mathbb{R}$ is so defined that 3

$$f(x) = \begin{cases} \frac{1}{7} & \text{when } x \text{ is rational} \\ \frac{1}{17} & \text{when } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at no point in $\mathbb{R}$.

(c) If a function $f(x)$ be defined as $f(x) = 0$ when $x \neq 0$ and $f(x) = 1$ when $x = 0$, prove that there exists no function $g(x)$ such that $g'(x) = f(x)$. 2

GROUP-E

Answer any *five* questions from the following 5×5 = 25

17. Define limit point of a subset of $\mathbb{R}\times\mathbb{R}$. Show that the set $Q\times Q = \{(x, y) \mid x, y$ are both rational numbers$\}$ is neither open nor closed in $\mathbb{R}\times\mathbb{R}$. 1+4

18. Let $f: S \to \mathbb{R}$ be a function, where $S \subset \mathbb{R}^2$. If f is continuous at a point $(a, b) \in S$ then show that $f(x, b)$ is continuous at $x = a$ and $f(a, y)$ is continuous at $y = b$. Is the converse of the result true? Justify your answer. 5

19. Let $f: S \to \mathbb{R}$ be a function, where $S \subset \mathbb{R}^2$. What do you mean by differentiability of f at a point $(a, b) \in S$? Show that differentiability of f at (a, b) implies the continuity of f at (a, b) and the existence of first order partial derivatives at that point. 5

20. Let $f(x, y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \cos\frac{1}{y} & , \text{ when } x \neq 0,\ y \neq 0 \\ x^2 \sin\frac{1}{x} & , \text{ when } x \neq 0 \\ y^2 \cos\frac{1}{y} & , \text{ when } y \neq 0 \\ 0 & , \text{ when } x^2 + y^2 = 0 \end{cases}$ 3+2

Prove that both f_x and f_y exist at (0, 0) but none is continuous there. Examine the differentiability of $f(x, y)$ at (0, 0).

21. If u be a homogeneous function of x, y, z of degree n having continuous second order partial derivatives and if $u = f(\xi, \eta, \zeta)$ where ξ, η, ζ are the partial derivatives of u w.r.t. x, y, z respectively, prove that 5

$$\xi \frac{\partial u}{\partial \xi} + \eta \frac{\partial u}{\partial \eta} + \zeta \frac{\partial u}{\partial \zeta} = \frac{nu}{n-1},\ (n \neq 1)$$

22. If $f(x, y)$ is a function of two variables x and y such that first order partial derivatives f_x and f_y are differentiable at an interior point (a, b) of the domain of definition of the function then show that $f_{xy}(a, b) = f_{yx}(a, b)$. 5

23. Let u, v be functions of α, β, γ having continuous first order partial derivatives and α, β, γ be functions of x and y having continuous first order partial derivatives. Prove that 5

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\alpha,\beta)} \cdot \frac{\partial(\alpha,\beta)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\beta,\gamma)} \cdot \frac{\partial(\beta,\gamma)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(\gamma,\alpha)} \cdot \frac{\partial(\gamma,\alpha)}{\partial(x,y)}$$

24. Write the conditions so that the functional equation $f(x, y) = 0$ does define an implicit function. Show that the equation $y^2 - yx^2 - 2x^5 = 0$ determine uniquely implicit function in the neighbourhood of the point (1, –1). Also find the first order derivative of the solution. 2+2+1

25. Find the condition, by using Jacobian, that the expression $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$ can be expressed as product of two linear factors. 5

GROUP-F

Answer any *two* questions from the following 5×2 = 10

26. Find the area between the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. 5

27. Find the moment of inertia of a thin uniform lamina in the form of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis. 5

28. Determine the co-ordinates of the centre of gravity of a segment of the parabola $y^2 = ax$ cut off by the straight line $x = a$. 5

29. Find the volume of the solid generated by the revolution about x-axis, of two arcs intercepted by the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 9a^2$. 5

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within* 1 *hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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