WEST BENGAL STATE UNIVERSITY

B.Sc. Honours Part-III Examination, 2022

MATHEMATICS

PAPER: MTMA-V

Time Allotted: 4 Hours | Full Marks: 100

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

GROUP-A

(Marks-70)

Answer Question No. 1 and any *five* questions from the rest

1. Answer any ***five*** questions from the following: $3\times5=15$

(a) Correct or justify: If $S=(-1,\,1)$ and $T=\{n\,|\,n\in\mathbb{Z}$ and $-m\le n\le m$, for some fixed integer $m>0\}$ then $S\cup T$ is compact.

(b) If $f:[a,\,b]\longrightarrow\mathbb{R}$ be continuous in $[a,\,b]$, $f(x)>0$ and

$F(x)=\int\limits_a^x f(t)\,dt$, $a\le x\le b$; then prove that F is strictly increasing in $[a\,,\,b]$.

(c) Show that the arc of the upper half of the cardioid $r=a(1-\cos\theta)$ is bisected at $\theta=2\pi/3$.

(d) Test the convergence of $\int\limits_0^1\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx$, where $k^2<1$.

(e) Show that f is not Riemann integrable on [0, 1] where,

$$f(x)=\begin{cases}x\ ; & x\in\mathbb{Q}\cap[0,\,1]\\ 0\ ; & x\notin\mathbb{Q},\,x\in[0,\,1]\end{cases}$$

(f) If $f(x)=x[x],\ 0\le x\le 4$, show that f is a function of bounded variation and find the total variation of f over [0, 4].

(g) Show that the function $\log x=\int\limits_1^x\frac{1}{t}\,dt,\ x>0$ is differentiable on $(0,\,\infty)$ and $\frac{d}{dx}(\log x)=\frac{1}{x}$, for all $x>0$.

(h) Show that the series $1+\sum\limits_{n=1}^{\infty}\frac{e^{-2nx}}{4n^2-1}$ is uniformly convergent on $[0,\,\infty)$.

(i) Evaluate $\iint_R xy(x^2+y^2)\,dxdy$, where R is the rectangle $[0,\ a]\times[0,\ b]$ in $\mathbb{R}^2$.

2. (a) Prove that a compact subset of $\mathbb{R}$ is closed and bounded in $\mathbb{R}$. 4

(b) If $f: D\to\mathbb{R}$ be a continuous function on a compact subset D of $\mathbb{R}$, then prove that $f(D)$ is compact in $\mathbb{R}$. 4

(c) If $G=\left\{\left(\frac{x}{2},\ \frac{1}{2}(x+1)\right): x\in(0,\ 1)\right\}$ then show that G is an open cover of (0, 1) but it has no finite sub cover for (0, 1). 3

3. (a) If $f: V\to\mathbb{R}$ is an uniformly continuous function, $V\subset\mathbb{R}$ and $\{x_n\}$ is a Cauchy sequence in V, show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. 3

(b) Let $f_n(x)=\dfrac{nx}{1+n^2x^2}$, $x\in(0,\ \infty)$, $n\in\mathbb{N}$. Show that the sequence $\{f_n\}$ converges pointwise but not uniformly. 2+2

(c) If $\{f_n\}$ is a sequence of continuous functions defined on an interval $[a,\ b]$ converging uniformly to a function f, show that f is continuous on $[a,\ b]$. 4

4. (a) Let a be the only point of infinite discontinuity of the functions f and g which are both integrable on $[a+\varepsilon,\ b]$ for all ε satisfying $0<\varepsilon<b-a$ and $f(x)>0,\ g(x)>0,\ \forall\ x\in(a,\ b]$. 4

If $\lim\limits_{x\to a+}\dfrac{f(x)}{g(x)}=l$, where l is a non-zero finite number, then prove that $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge or diverge together.

(b) Find the value of α for which $\int_0^\infty \dfrac{x^{\alpha-1}\log x}{1+x}\,dx$ will converge. 4

(c) Using Dirichlet's test, show that $\sum\limits_{n=1}^{\infty}\dfrac{\cos nx}{n}$ is a uniformly convergent series on any closed interval $[a,\ b]$ contained in $(0,\ 2\pi)$. 3

5. (a) Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$, $f:[0,\ 1]\to\mathbb{R}$ is defined by, 4

$$f(x)=a_n\ ,\quad \frac{1}{n+1}<x\le\frac{1}{n}\ ,\quad n\in\mathbb{N}$$
$$\quad =0\ ,\quad x=0$$

Show that f is R-integrable.

(b) Let $f:[a,\ b]\to\mathbb{R}$ be R-integrable, define $F:[a,\ b]\to\mathbb{R}$ by $F(x)=\int_a^x f(t)\,dt$. Show that F is continuous on $[a,\ b]$. State a sufficient condition for differentiability of F in $(a,\ b)$. 3+1

(c) Find the values of p, if any, that the integral $\int_1^\infty \frac{dx}{x^p}$ is convergent. 3

6. (a) For each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ is a continuous function on $D \subset \mathbb{R}$. If the series $\sum f_n$ is uniformly convergent on D then prove that the sum function is continuous on D. 4

(b) Using Abel's test show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^p (1+x^n)}$ converges uniformly for all $p > 0$ on [0, 1]. 4

(c) Correct or justify : If $\sum_{n=0}^{\infty} |a_n|$ is convergent then $\int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$. 3

7. (a) If $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence 1 and $\sum_{n=0}^{\infty} a_n$ is convergent then prove that $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on [0, 1]. 5

(b) Show that $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ may be integrated term-by-term from 0 to x, $-1 < x < 1$ and thus prove that 4+2

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\cdots \quad (-1 < x < 1)$$

Deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

8. (a) If $f : [a, b] \to \mathbb{R}$ be Riemann integrable, then prove that 3

$F(x) = \int_a^x f(t)\, dt$ is of bounded variation over $[a, b]$.

(b) A function $f : [0, 1] \to \mathbb{R}$ is defined by 3

$$f(x) = \sin \frac{\pi}{x} \quad , \quad 0 < x \le 1$$
$$= 0 \quad , \quad x = 0$$

Is the function f of bounded variation over [0, 1]?

(c) If $f(x) = \{\pi - |x|\}^2$ on $[-\pi, \pi]$, obtain the Fourier series of f. 5

Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

9. (a) Expand $f(x, y) = e^x \sin y$ about the point $(0, \frac{\pi}{2})$, calculate up to the second degree terms. 4

(b) Find the stationary points and the extreme values of the function $f(x, y) = x^3 + 3x^2 + y^2 + 4xy$. 3

(c) Find the volume of the cone whose base is the ellipse $4x^2 + 9y^2 = 36$ and the vertex is at (0, 0, 4). 4

10.(a) Find the Fourier series of the function *f* defined by, 5+1

$$f(x) = \frac{2x}{\pi} + 1, \quad -\pi \le x < 0$$
$$= \frac{2x}{\pi} - 1, \quad 0 \le x \le \pi$$

Find the sum of the series at the points $x = \pi$ and $x = -\pi$.

(b) Show that $\int_0^\infty \frac{\sin bx}{x(a^2 + x^2)} dx = \frac{\pi}{2a^2}(1 - e^{-ab})$, where $a > 0,\ b \ge 0$. 5

GROUP-B

(Marks-15)

Answer any *one* question from the following

11.(a) If (*X*, *d*) is a metric space show that the function $\rho : X \times X \to \mathbb{R}$, defined by $\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$, $x, y \in X$, is also a metric on *X*. 5

(b) Show that every set in a discrete metric space is an open set as well as a closed set. 4

(c) Define (i) a bounded sequence (ii) a Cauchy sequence in a metric space. Show that in a metric space a Cauchy sequence is bounded. Does the converse hold? Support your answer. 1+1+3+1

12.(a) Let $C[a,b]$ denote the set of all real valued continuous functions defined on the closed interval $[a, b]$. Define $d : C[a, b] \times C[a, b] \to \mathbb{R}$ by 5

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)| \text{ for all } f, g \in C[a, b].$$

Show that *d* is a metric on *C*[*a*, *b*].

(b) Let (*X*, *d*) be a metric space. Show that for any $x, y \in X$, $x \ne y$ there are open balls B_1 and B_2 in (*X*, *d*) such that $x \in B_1$, $y \in B_2$, $B_1 \cap B_2 = \Phi$. Also show that $\{x\}$ is closed in $\{X, d\}$ for any $x \in X$. 2+2

(c) Show that in a discrete metric space a convergent sequence is eventually constant. 2

(d) Let (*X*, *d*) be a metric space and $A \subset X$. Show that a point $p \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in *A* which converges to *p*, ($\overline{A}$ denotes the closure of *A*). 4

GROUP-C

(Marks-15)

Answer any *one* question from the following

13.(a) A point $(x_1, x_2, x_3) \neq (0, 0, 1)$ in unit sphere has been mapped under stereographic projection to a point $z = x + iy$ in the complex plane, where the complex plane passes through the equator of the sphere. Find x, y in terms of x_1, x_2, x_3. 4

(b) If $f: G \to \mathbb{C}$ where $f(x+iy) = u(x, y) + iv(x, y)$ be a function of a complex variable on a region G such that $u(x, y), v(x, y)$ are differentiable at (x_0, y_0) and the Cauchy-Riemann equations are satisfied at (x_0, y_0), then prove that f is differentiable at $z_0 = x_0 + iy_0$. 6

(c) Show that, 5

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, & z = x + iy \neq 0 \\ 0, & \text{for } z = 0 \end{cases}$$

satisfies Cauchy-Riemann equations but is not differentiable at $z = 0$.

14.(a) If the complex sequence $\{z_n\}$ where $z_n = a_n + ib_n$, for $n \in \mathbb{N}$ converges to $z = a + ib$ then prove that $\{|z_n|\}$ converges to $|z|$. Is the converse of the above result true? — Justify your answer. 2+2

(b) Show that if a function $f: \mathbb{C} \to \mathbb{C}$ is differentiable at $z_0 \in \mathbb{C}$ then it is continuous there. Is the converse true? Support your answer. 2+2

(c) Given that $f(z) = u(x, y) + iv(x, y)$ is analytic and u, v have continuous second order partial derivatives. Prove that u, v are harmonic functions. 3

(d) Show that $u(x, y) = e^x(x \cos y - y \sin y)$, $x + iy \in \mathbb{C}$ is a harmonic function. Find a conjugate harmonic function of u. 4

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within* 1 *hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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