WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 5th Semester Examination, 2021-22

MTMACOR12T-MATHEMATICS (CC12)

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times 5 = 10$

(a) Let G be a group. If the mapping $\alpha : G \to G$ defined by $\alpha(g) = g^{-1}$, for all $g \in G$ is an automorphism of G, prove that G is an Abelian group.

(b) Let G be a group and $x, y, z \in G$. Prove that $[xy, z] = y^{-1}[x, z]\, y[y, z]$. (The notation $[a, b]$ stands for the commutator of elements a, b in G.)

(c) Let $(\alpha, \beta) \in \mathbb{Z}_{18}\times S_5$, where $\alpha = [2] \in \mathbb{Z}_{18}$ and $\beta = (1\quad 3)(2\quad 5\quad 4) \in S_5$. Find the order of (α, β) in the external direct product $\mathbb{Z}_{18}\times S_5$ of the additive group $\mathbb{Z}_{18}$ and symmetric group S_5.

(d) Show that the external direct product $\mathbb{Z} \times \mathbb{Z}$ of the additive group $\mathbb{Z}$ of integers with itself is not a cyclic group.

(e) Show that every Abelian group of order 45 has an element of order 15.

(f) For a prime p, prove that every group of order $p^n (n > 0)$ contains a normal subgroup of order p.

(g) Let G be a group that acts on a nonempty set S. Prove that, for any nonempty subset T of S, the set $\text{Fix}_G(T) = \{g \in G : gx = x, \forall\, x \in T\}$ is a subgroup of G.

(h) Prove that a finite group of order 28 contains a subgroup of order 14.

(i) Show that no group of order 74 is a simple group.

2. (a) Let G be a finite group with identity e. Suppose that G has an automorphism α which satisfies the condition 'for all $x \in G$, $\alpha(x) = x \Rightarrow x - e$'. 2+2

(i) Prove that, for every $g \in G$, there exists $x \in G$ such that $g = x^{-1}\alpha(x)$.

(ii) If α is of order 2 in the automorphism group of G, then show that the group G is Abelian.

(b) Let G be an infinite cyclic group. Prove that the group of automorphism of G is isomorphic to the additive group $\mathbb{Z}_2$ of integers modulo 2. 4

3. (a) Show that the commutator subgroup G' of a group G is a normal subgroup of G. 3

(b) Let H be a subgroup of a group G. Prove that $H \subseteq G'$ if and only if H is a normal subgroup of G and the factor group G/H is Abelian. 5

4. (a) Define internal direct product of two subgroups of a group. 1

(b) Two subgroups H and K of a group G are such that $G = HK$ and $H \bigcap K = \{e\}$, where e is the identity in G. Prove that G is an internal direct product of H and K if and only if the subgroups H and K are normal in G. 4

(c) If G is an internal direct product of two of its subgroups H and K, prove that $G/H \simeq K$. 3

5. (a) Let G be an Abelian group of order 8. Suppose that G contains an element a such that $o(a) = 4$ and $o(a) \geq o(b)$ for all $b \in G$. Prove that G is isomorphic to the external direct product $\mathbb{Z}_4 \times \mathbb{Z}_2$ of the additive groups $\mathbb{Z}_4$ and $\mathbb{Z}_2$. 4

(b) Find the number of elements of order 5 in the external direct product $\mathbb{Z}_{15} \times \mathbb{Z}_5$ of the groups $\mathbb{Z}_{15}$ and $\mathbb{Z}_5$. 4

6. (a) Let G be a non-cyclic group of order p^2. Then show that $G \simeq z_p \oplus z_p$. 4

(b) Find all non-isomorphic Abelian groups of order 16. 4

7. (a) Let G be a finite group and A be a G-set. Then for each $a \in A$, show that $|\text{Orb}(a)| = [G : G_a]$, where $\text{Orb}(a)$ denotes the orbit of a in A and G_a is the stabilizer of a in G. 5

(b) Using the result stated in (a), prove that every action of a group of order 39 on a set of 11 elements has a fixed element. 3

8. (a) Let G be a p-group for a prime p. If A is a finite G-set and $A_0 = \{a \in A : ga = a$ for all $g \in G\}$, then prove that $|A| \equiv |A_0| (\text{mod } p)$. 4

(b) Let G be a finite group and H be a subgroup of G of index n such that $|G|$ does not divide $n!$. Then show that G contains a non-trivial normal subgroup. 4

9. (a) Is there any group of order 15 whose class equation is given by
$15 = 1+1+1+1+3+3+5$? Justify your answer. 2

(b) Write down the class equation of S_4. 3

(c) Prove that a subgroup H of a group G is a normal subgroup if and only if H is a union of some conjugacy classes of G. 3

10. (a) Determine all the Sylow 3-subgroups of the alternating group A_4. 3

(b) Show that every group of order 147 has a normal subgroup of order 49. 2

(c) For any prime p, prove that every group of order p^2 is commutative. 3

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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