WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 5th Semester Examination, 2020, held in 2021

MTMADSE03T-MATHEMATICS (DSE1/2)

PROBABILITY AND STATISTICS

Time Allotted: 2 Hours — Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* questions from the rest

1. Answer any ***five*** questions from the following: 2×5 = 10

(a) If A and B are two events such that $P(A)=P(B)=1$, then show that $P(A+B)=1$.

(b) If A and B are independent events then prove that $\overline{A}$ and B are independent.

(c) Find the value of the constant k, so that the function $f(x)$ defined below:

$$f(x)=\begin{cases} x & , \quad 0<x\le 1 \\ k-x & , \quad 1<x\le 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

is a probability density function.

(d) If X is a symmetric binomial variable with mean 16. Find the standard deviation.

(e) If a random variable X follows Poisson distribution satisfying $2P(X=0)=P(X=1)$, then find $P(X>0)$.

(f) If the regression lines of two random variables X and Y are $3y=4x+1$ and $3y-x=7$, then find the means of X and Y.

(g) The joint density function of two random variables X and Y is

$$f(x,\ y)=\begin{cases} 2 & , \quad 0<x<y<1 \\ 0 & , \quad \text{elsewhere} \end{cases},$$

then find the conditional density function $f_x(x/y)$ of X given $Y=y$.

(h) Define Markov chain and steady state condition.

(i) Show by Tchebycheff's inequality that in 1000 throws with a coin the probability that the number of heads lies between 400 and 600 is at least $\frac{39}{40}$.

(j) Explain the terms Null Hypothesis and Alternative Hypothesis.

2. (a) Give the axiomatic definition of Probability. Use this to prove 1+2+2

(i) $0\le P(A)\le 1$ for any event A,

(ii) If $A\subset B$, then prove that $P(A)\le P(B)$.

(b) If $\{A_n\}$ be monotonically increasing sequence of events then prove that $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n)$. 3

3. (a) Define the conditional probability of the event A on the hypothesis that the event B has occurred. Show that it satisfies all the axioms of probability. 1+3

(b) A secretary writes four letters and the corresponding addresses on envelopes. If he inserts the letters in the envelopes at random irrespective of address, then calculate the probability that all the letters are wrongly placed. 4

4. (a) If $F(x) = \begin{cases} 0 & , \ -\infty < x < 0 \\ 1-e^{-x} & , \ 0 \le x < \infty \end{cases}$, show that $F(x)$ is a possible distribution function and find the density function. 1+2

(b) A discrete random variable X has the following probability mass function: 1+2+2

$x_i = i$	–3	–2	–1	0	1	2
$f_i = P(x = i)$	k	$2k$	$2k^2$	$3k^2$	k^2	$6k^2+8k$

(i) Determine the value of k.

(ii) Find the distribution function $F(x)$.

(iii) Evaluate $P(X < -1)$.

5. (a) Find the moment generating function of a binomial (n, p) variate X and from this find the variance. 1+2

(b) Obtain the recurrence relation $\mu_{k+1} = \lambda\left(k\mu_{k-1} + \frac{d\mu_k}{d\lambda}\right)$ for the Poisson distribution with parameter λ where μ_k is the k-th central moment. Hence find the standard deviation of the Poisson distribution. 3+2

6. (a) If the correlation coefficient of the random variables X and Y is $\rho(X, Y)$ then prove that $-1 \le \rho(X, Y) \le 1$. 3

(b) If X and Y are uncorrelated then prove that $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$, where $\text{Var}(X)$ implies variance of X. 2

(c) If, for any pair of correlated random variables X and Y, a linear transformation $(X, Y) \to (U, V)$ is given by $U = X\cos\alpha + Y\sin\alpha$, $V = -X\sin\alpha + Y\cos\alpha$ then U and V will be uncorrelated if $\tan 2\alpha = \frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$. 3

7. (a) Let the p.d.f. of the two dimensional random variable (X, Y) is a constant c, say, when $x^2 + y^2 < a^2$, otherwise it vanishes. Find (i) the value of c and (ii) the marginal distributions of X and Y. Check whether X and Y are independent. 2+2+1

(b) Find the characteristic function of a Normal (m, σ) random variable X. 3

8. (a) A random variable X has a density function $f(x)$ given by 3

$$f(x)=\begin{cases} e^{-x} & , \quad x\geq 0 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Show using Tchebycheff's inequality that $P(|X-1|\geq 2)\leq \frac{1}{4}$.

(b) If X_n is a binomial (n, p) variate, then prove that $\lim_{n\to\infty} P\left(\left|\frac{X_n}{n}-p\right|\geq \varepsilon\right)=0$. 3

(c) If $X_1, X_2, X_3, \cdots, X_n, \cdots$ be a sequence of mutually independent random variables each having Poisson-1 distribution, then prove that $\frac{X_1+X_2+\cdots+X_n}{n}$ is asymptotically normal $\left(1, \frac{1}{\sqrt{n}}\right)$. 2

9. (a) State weak law of large numbers and obtain Bernoulli's theorem as a particular case of weak law of large numbers. 1+3

(b) State strong law of large numbers and interpret the law statistically. 1+1

(c) Suppose $\{x_n\}$ is a Markov chain with 3 states and the transition probability matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$, show that all the states are ergodic. 2

10.(a) Show that the sample variance is not an unbiased estimate of the population variance. Hence find an unbiased estimate for the population variance. 4

(b) In a random sample of 400 articles 40 are found to be defective. Obtain the confidence interval for the true proportion of defectives in the population of such articles. 4

[Given $\int_0^{1.96} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0.4750$]

11.(a) Distinguish between sampling distribution and distribution of sample. 2+2

(b) It is required to estimate the mean of the normal population having a sample sufficiently large so that the probability will be 0.95 that the sample mean will not differ from the population mean by more than 25% of the population standard deviation. How large should be the sample? 4

12.(a) Define best linear unbiased estimate (BLUE) of a population parameter. Prove that for any population, sample mean $\bar{x}$ is best linear unbiased estimate of the population mean m, where the population standard deviation exists. 1+4

(b) Prove that an unbiased estimator A_n of an unknown population parameter θ is a consistent estimator of θ if $\lim_{n\to\infty} \text{var}(A_n)=0$ 3

13. Define critical region for testing a statistical hypothesis, and power of a test. The random variable X denoting the amount of consumption of a commodity follows the distribution: 2+2+2+2

$$f(x,\ \theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}},\ 0<x<\infty,\ \theta>0$$

The hypothesis $H_0:\theta=5$ is rejected in favour of $H_1:\theta=10$, if 15 units or more, chosen randomly be consumed. Obtain the size of the two types of errors and power of the test.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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