Mathematical Physics II

Numerical Methods using Python: Newton's method of Interpolation

- **What is interpolation and when do we need it**

  Let we have $n+1$ no. of data points $\{x_i, y_i\}$ for $i = 0$ to n. Now we want to get the value of y at a point x within the range $[x_0, x_n]$, but does not match with any one of x_i's. For this purpose we need to fit the data points with a polynomial $p(x)$. Once we able to do this we shall get the value of y for any value of x within the above mentioned range. Th polynomial $p(x)$ is called **interpolating polynomial** and the approximation $y = p(x)$ for any value of x in the interval is called **interpolation**. Here we shall discuss first Newton's Interpolation and then Newton-Gregory forward interpolation.

- **Newton's Interpolation:** The Newton's form of polynomial for $n+1$ number of data points $\{x_i, y_i\}$ is

  $$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ... + a_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

  So to get the interpolating polynimial we need to find the coefficients a_i's. Now at $x = x_0$, $y = y_0$; $\therefore p_n(x_0) = a_0 = y_0$

  Similarly at $x = x_1$,$y = y_1$; $\therefore p_n(x_1) = a_0 + a_1(x - x_0) = y_1$ and at $x = x_2$,$y = y_2$; $\therefore p_n(x_2) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) = y_2$ Using the value of a_0 we get

  $$a_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

  Similarly substituting a_0 and a_1 one get

  $$a_2 = \frac{1}{x_2 - x_0}\left(\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right)$$

  Let us consider the notation

  $$y[x_i] = y_i$$

  $$y[x_i, x_{i+1}] = \frac{y[x_{i+1}] - y[x_i]}{x_{i+1} - x_i}$$

  $$y[x_i, x_{i+1}, x_{i+2}] = \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

  Using this we can write

  $$a_0 = y[x_0]$$

  $$a_1 = y[x_0, x_1]$$

$$a_2 = y[x_0, x_1, x_2]$$

Thus

$$a_i = y[x_0, x_1, x_2, ...x_i]$$

$$\therefore p_n(x) = \sum_{i=1}^{n} y[x_0, ..., x_i] \prod_{j=0}^{i-1} (x - x_j)$$

This equation is called Newton's divided difference interpolation polynomial formula.

- **Gregory-Newton forward difference formula:**

  Let we have the function values for n number of equidistant points. If h be the stepsize then ith point will be $x_i = x_0 + ih$. Let the first forward difference is represented as

  $$\Delta y_i = y_{i+1} - y_i$$

  Similarly second forward difference is

  $$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

  Therefore the general representation for ith order forward difference is

  $$\Delta^i y_i = \Delta^{i-1} y_{i+1} - \Delta^{i-1} y_i$$

  Now we shall use this representation in the Newton's divided difference interpolation polynomial formula. As in this case for any i

  $$x_{i+1} - x_i = h$$

  So

  $$y[x_0, x_1] = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$
  $$\therefore \Delta y_0 = hy[x_0, x_1]$$

  Similarly

  $$\Delta y_1 = hy[x_1, x_2]$$

  Now

  $$\begin{aligned} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 &= h(y[x_1, x_2] - y[x_0, x_1]) \\ &= h(x_2 - x_0) y[x_0, x_1, x_2] \\ &= h.2h.y[x_0, x_1, x_2] \\ &= 2h^2 y[x_0, x_1, x_2] \end{aligned}$$

  By the method of induction one can write

  $$\Delta^i y_0 = i! h^i y[x_0, x_1, x_2, ..., x_i]$$

  $$\therefore y[x_0, x_1, x_2, ..., x_i] = \frac{\Delta^i y_0}{i! h^i}$$

Therefore Newton's divided difference interpolation polynomial form reduces to

$$p_n(x) = \sum_{i=0}^{n} \frac{\Delta^i y_0}{i!h^i} \prod_{j=0}^{i-1} (x - x_j)$$

Let the value of x at which we want to get y is given by

$$x = x_0 + sh$$

$$\therefore x - x_j = (s - j)h$$

as $x_j = x_0 + jh$.
Therefore the Newton's divided difference interpolation polynomial form finally becomes:

$$p_n(x) = \sum_{i=0}^{n} \frac{\Delta^i y_0}{i!h^i} \prod_{j=0}^{i-1} (s - j)h$$

$$= \sum_{i=0}^{n} \frac{\Delta^i y_0}{i!h^i} [s(s-1)(s-2)...(s-i+1)]h^i$$

$$= \sum_{i=0}^{n} \Delta^i y_0^s C_i$$

This is known as Gregory-Newton forward difference formula.

References

1. 'Numerical Methods', E. Balagurusamy, TMH, 2016.
2. 'An Introduction to Numerical Analysis', Devi Prasad, Narosa, Third Edition, 2012.

- **Example**: In the picture one sample python program using Gregory-Newton forward difference formula has been shown. Here we have to find out the interpolation polynomial for the following data set so that for any value of x within the given range we can get y.

x	0	0.2	0.4	0.6	0.8
y	0.12	0.46	0.74	0.9	1.2

```python
#!/usr/bin/env python2
# -*- coding: utf-8 -*-
"""
Created on Sun May 26 10:34:00 2019

@author: sonali
"""
import matplotlib.pyplot as plt
import numpy as np
n=5
x=[0, 0.2, 0.4, 0.6, 0.8]
y=[0.12,0.46,0.74,0.9,1.2]
z=y
xx=np.arange(0,0.9,0.01)
f=[]
for ii in range(len(xx)):
    y=z
    p=y[0]
    a=[]
    h=x[1]-x[0]
    s0=(xx[ii]-x[0])/h
    s=1
    a.append(y[0])
    for i in range(n-1):
        delf=[]
        for j in range(n-1-i):
            delf.append(y[j+1]-y[j])
        s=s*(s0-i)/(i+1)
        p=p+s*delf[0]
        y=delf
        a.append(y[0])
    print (xx[ii],p)
    f.append(p)
plt.plot(xx,f)
plt.plot(x,z,'X')

```

Figure 1: Python Code for Gregory-Newton Forward Rule of Interpolation

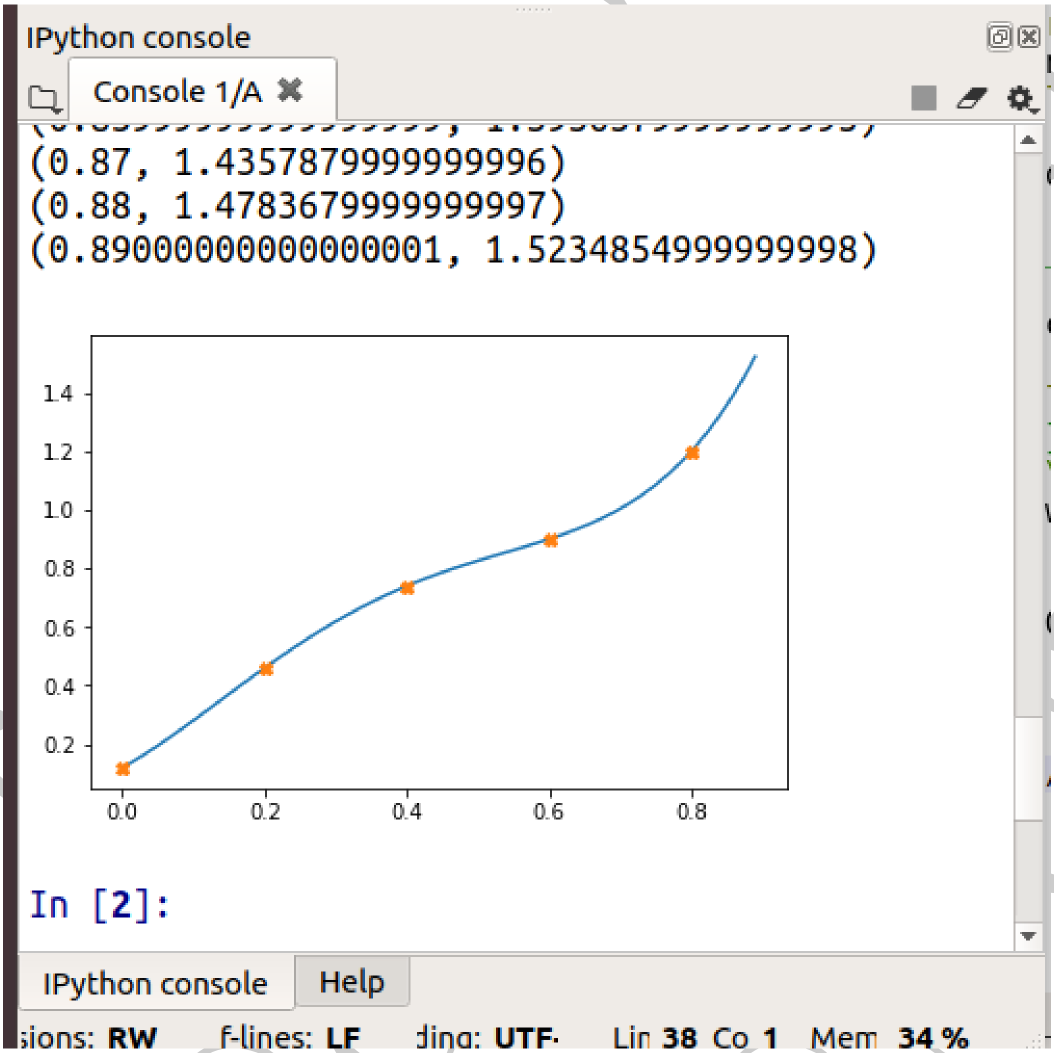

Figure 2: Fitted curve along with data points