Field Emission of High Energy Electrons in $1+1$-Dimension from the Strongly Magnetized Neutron Star Poles

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Date of Submission: 31st March, 2016 Date of Acceptance: 24th April, 2016

Abstract

A formalism for field emission of relativistic electrons in $1+1$-dimension relevant for strongly magnetized neutron stars / magnetars is developed. In this scenario it has been noticed that the introduction of driving potential in Dirac equation following the conventional form of relativistic quantum hadrodynamic approach, the quantum tunneling or popularly known as the Fowler-Nordheim cold emission of electrons is completely forbidden with both the scalar type or a vector type surface barrier.

Key words: Fowler-Nordheim emission, Field emission, Dirac equation, Quantum tunneling, Quantum hadrodynamics

1. Introduction

There are mainly three kinds of electron emission processes from metal surfaces, they are: (i) thermionic emission, (ii) photoelectric emission and (iii) cold emission or field emission.

The field emission or cold emission process, for which a relativistic formalism in $1+1$-dimension is developed in this article has relevance in electron emission from the polar region of strongly magnetized neutron stars or magnetars, induced by strong external electric field. It is well known that this strong electric field at the polar region is produced by the rotating pulsar magnetic field. The temperature of such compact objects are extremely low, less than the corresponding electron Fermi temperature, therefore one can safely assume it to be zero.

Field emission can also happen from solid and liquid surfaces, or from individual atoms. It has been noticed from condensed matter experimental results, that the field emission from metals occur in presence of high electric field; the gradients are typically higher than 1000 volts per micron and strongly dependent upon the work function of the

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material. Unlike the thermionic emission and photo-emission of electrons, the field emission process can only be explained by quantum tunneling of electrons. It has no counter classical explanation. Further, for a general type of surface barrier, this purely quantum mechanical problem can not be solved exactly, a semi-classical approximation, known as WKB is needed to get tunneling coefficients. It is generally expected from the concept of statistical mechanics for finite /semi-finite system, that because of quantum fluctuation, electrons from the sea of conduction electrons (degenerate electron gas) always try to tunnel out through the metal surface (surface barrier). However, as soon as an electron comes out, it induces an image charge on the metal surface, which pulls the electron back to the metal body and does not allow it to move far away in the atomic scale. On the other hand, if a strong attractive electrostatic field is applied near the metal surface, then depending on the Fermi energy of electrons, the local work function of the metal and the height of the surface barrier, the emitted electrons may then overcome the effect of image charges and get liberated. Since the external strong electric field is causing such emission and does not depend on the thermal properties of the metal, even the metal can be at zero temperature, it is called the field emission or the cold emission.

The theory of field emission from bulk metals was originally proposed by Fowler and Nordheim in an Nobel winning work in the proceedings of Royal Society of London in the year 1928[1] (see also[2;3;4;5] for some interesting discussions and possible applications in the field of cold emission). Fowler and Nordheim tunneling is the wave-mechanical tunneling of electrons through a triangular type barrier produced at the surface of an electron conductor by applying a very high electrostatic potential linearly changing with distance.

Further the exactly solvable models with simple type tunneling barrier lead to equations[1;2] that underestimates the emission current density by a factor of 1000 or more. If a more realistic type barrier model is used by inserting an exact potential at the surface, even for the simplest form of the Schrödinger equation, a complicated mathematical problem arises over the resulting differential equation. It is in principle therefore mathematically impossible to solve the equation exactly in terms of the usual functions of mathematical physics, or in any simple manner.

The cold emission not only has relevant in human laboratory but equally important in the case of astro-physical objects, e.g., strongly magnetized neutron stars, also known as magnetars. In a recent work[6] we have developed a formalism for cold emission of electrons from the polar region of strongly magnetized neutron stars, when Landau levels for the electrons are populated because of the quantizing nature of strong magnetic field. In the present article we shall study the cold emission of high energy electrons in $1+1$-dimension, which, in particular has importance in the case of field emission of electrons from the polar region of neutron stars with ultra-strong magnetic field. To the best of our knowledge, a relativistic version of cold emission model in $1+1$-dimension, even with a simple type potential barrier has not been reported in the literature. Since the strength of magnetic field for magnetars, particularly at the polar regions of such objects are extremely high one may assume without appreciable error that the electrons move along the field lines. As a consequence we may assume that the high energy electrons near the poles obey one dimensional form of Dirac equation.

Based on the above discussion, the aim of this article is to develop a formalism for the field emission of relativistic electrons in $1+1$-dimension. We have organized the article in the following manner: In section-2 we have studied transmission coefficient for electron emission with a scalar type potential barrier. In section-3 we have repeated the same calculation, replacing the scalar potential by a vector type potential barrier. In this section, to have an analytical solution for transmission coefficient, we have first incorporated the so called zeroth order approximation. In section-4, following first order iterative technique, which is the next order better approximation, we have obtained transmission coefficient for electron emission with the vector type potential barrier. In the last section we have given the conclusions.

2. Relativistic Emission Model in Presence of Scalar Potential Barrier

In the original paper by Fowler and Nordheim on cold emission, the non-relativistic Schrödinger equation was used[1]. In the present relativistic version we shall replace it with Dirac equation for the high energy electrons. Here we have introduced the triangular type potential barrier at the polar region of the strongly magnetized neutron star, following the conventional approach of relativistic quantum hadro-dynamics[7]. We have adopted the field theoretic technique to avoid Klein paradox[8]. In the simplified picture of quantum hadro-dynamic model the interaction can either be of scalar type or vector type or in general both type. In this model the usual form of Dirac equation with both scalar and vector type interactions is given by[9;10]

$$[\gamma^\mu (i\partial_\mu - g_v V_\mu) - (m - g_s \phi)]\psi = 0 \tag{1}$$

where g_v and g_s are respectively the vector and scalar coupling constants, V_μ and ϕ are the corresponding fields and γ_μs are the Dirac γ-matrices. In $1+1$-dimensional model, assuming the motion of electrons along x-direction, the corresponding time independent Dirac equation will also be reduced to one-dimensional form. Further, we replace the scalar part and the vector part of the fields by the non-relativistic triangular shape potentials at the interface between matter and vacuum. The electrons immediately after emission from the crustal matter near the poles travel in triangular shape potential, whereas, electrons within the crustal matter, just before the emission near the polar region moving freely along x-axis. In the case of vector potential, only the zeroth part is retained, whereas its spacial part vanishes because of rotational symmetry[7]. In one-dimension, for the sake of convenience we use the following form of Dirac equation in presence of a scalar potential $V(x)$.

$$[\vec{\alpha}.\vec{p} + \beta(m + V(x))]\psi = E\psi \tag{2}$$

where $\vec{\alpha}$ and β are the usual Dirac matrices. Following Fowler and Nordheim[1], the triangular shape potential barrier is represented by the form $V(x) = C - eFx$, where C is the surface barrier and F (absorbing the electron charge, we express eF by F) is the constant electric field, acts as the driving force for electron emission and E is the energy eigen value. The electric field F at the polar region of magnetized neutron star or magnetar is produced by the magnetic field of rotating neutron star and is approximately given by $F \sim 2 \times 10^8 P^{-1} B_{12} V cm^{-1}$, which is parallel to B at the poles[11]. Here P is the time period of the neutron star in second and B_{12} is the measure of magnetic field strength in units of 10^{12}. Now in $1+1$-dimension, the normal four-component Dirac equation reduces to a two component one. Then following the representation of[12;13] (see also[14] for further discussion), we have

$$\alpha = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

Defining $m^* = m + C$ as the effective electron mass and with the spinor notation

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \quad (4)$$

we have

$$-\frac{dv}{dx} + (m^* - Fx)v = Eu \quad (5)$$
$$\frac{du}{dx} + (m^* - Fx)u = Ev \quad (6)$$

After eliminating v from eqns.(5) and (6), we have

$$-\frac{d^2u}{dx^2} + Fu + (m^* - Fx)^2 u = E^2 u \quad (7)$$

To solve this equation analytically, we make the following transformation of coordinate:

$$\xi = F^{1/2}\left(\frac{m^*}{F} - x\right) \quad (8)$$

Then we have

$$-\frac{d^2u}{d\xi^2} + \xi^2 u = \left(\frac{E^2}{F} - 1\right)u \quad (9)$$

From which we get

$$\frac{d^2u}{d\xi^2} + \left\{\left(1 - \frac{E^2}{F}\right) - \xi^2\right\}u = 0$$

Defining $\lambda = 1 - E^2/F$, this equation may be re-arranged in the following form

$$\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)u = 0 \quad (10)$$

which is the well known form of differential equation satisfied by one dimensional quantum mechanical harmonic oscillator. With $\lambda = 1 + 2\nu$, we have $E^2 = 2(\nu + 1)F$, and the solution is given by

$$u = N\frac{E}{F^{1/2}}\exp\left(-\frac{\xi^2}{2}\right)H_\nu(\xi) = u_I \quad \text{(say)} \quad (11)$$

where N is the normalization constant and $H_\nu(\xi)$ is the Hermite polynomial of order ν. This spinor solution u is for those electrons of the crustal matter, which have already been liberated out through the surface barrier at the poles of strongly magnetized neutron star, into the magnetosphere produced by the liberated electrons and the produced electron-positron pairs under the influence of electric field F and magnetic field of strength B respectively.

Now the one-dimensional form of free Dirac equation for the electrons, which are within the crustal matter of the strongly magnetized neutron star, bounded by the scalar type surface barrier, is given by

$$(\vec{\alpha}.\vec{p} + \beta m)\psi = E\psi \quad (12)$$

This equation is satisfied by the electrons prior to emission from the crustal matter near the poles. Then using the same representation for the Dirac matrices as given in eqn.(3) and following the same algebraic technique as discussed before, we have from the above equation

$$\frac{du}{dx} + mu = Ev$$
$$-\frac{dv}{dx} + mv = Eu$$

Hence eliminating v we get

$$\frac{d^2u}{dx^2} + w_k^2 u = 0 \tag{13}$$

where $w_k = (E^2 - m^2)^{1/2}$ is the free electron momentum of energy eigen value E. Now following the notation of Fowler and Nordheim[1], we express the solution for free electrons within the crustal matter, confined by the surface barrier $V(x)$ near the poles, in the form

$$u = \frac{1}{w_k^{1/2}} \left[a \exp(iw_k x) + a' \exp(-iw_k x)\right] = u_{II} \quad \text{(say)} \tag{14}$$

where a is the probability amplitude for electrons moving along the positive direction of x (incident part), whereas a' is the corresponding quantity for left moving waves (reflected part from the surface barrier). Assuming the interface between the crustal matter and the magnetosphere is at $x = 0$, the wave function and their derivatives must be continuous at $x = 0$[1], i.e.,

$$u_{II}(0) = u_I(0) \quad \text{and} \quad u'_{II}(0) = u'_I(0) \tag{15}$$

Using the relation $H'_\nu(\xi) = 2\nu H_{\nu-1}(\xi)$, appears in the first derivative of u_I with respect to x, we have

$$a + a' = N w_k^{1/2} \frac{E}{F^{1/2}} \exp\left(-\frac{\xi_0^2}{2}\right) H_\nu(\xi_0) \tag{16}$$

and

$$i w_k^{1/2}(a - a') = NE \exp\left(-\frac{\xi_0^2}{2}\right) \left[\xi_0 H_\nu(\xi_0) - 2\nu H_{\nu-1}(\xi_0)\right] \tag{17}$$

where $\xi_0 = \xi(x = 0) = m^*/F^{1/2}$. These two conditions (eqns.(16) and (17)) may be rearranged in the following form

$$a + a' = X \quad \text{and} \quad a - a' = iY \tag{18}$$

where X and Y are two real quantities, given by

$$X = N w_k^{1/2} \frac{E}{F^{1/2}} \exp\left(-\frac{\xi_0^2}{2}\right) H_\nu(\xi_0)$$

$$Y = \frac{NE \exp\left(-\frac{\xi_0^2}{2}\right) \left[\xi_0 H_\nu(\xi_0) - 2\nu H_{\nu-1}(\xi_0)\right]}{i w_k^{1/2}}$$

Hence it is straight forward to verify that the transmission coefficient, defined by

$$T_R = 1 - \frac{|a'|^2}{|a|^2}, \tag{19}$$

vanishes exactly. From this analysis we may therefore conclude that if the barrier in combination with the external electrostatic driving force behaves like a scalar type potential and is triangular in shape at the surface, then electrons can not tunnel out through the surface barrier whatever be their kinetic energies and the strength of external electric field.

As an alternative, we have obtained the electron transmission current through the barrier in presence of the scalar potential $V(x)$ at the poles of strongly magnetars. As a first step we have obtained the solution for v from eqns.(5) and (6) and then using the well-known relation $H_{\nu+1}(\xi) = 2\xi H_\nu(\xi) - 2\nu H_{\nu-1}(\xi)$, the Dirac spinor is given by

$$\psi(\xi) = N \begin{pmatrix} \frac{E}{F^{1/2}} H_\nu(\xi) \\ H_{\nu+1}(\xi) \end{pmatrix} \exp\left(-\frac{\xi^2}{2}\right) \tag{20}$$

The transmission current in $1+1$-dimension is defined as

$$J_{trans} = \psi^\dagger \alpha \psi = i\psi^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi \tag{21}$$

Substituting ψ and $\psi^\dagger$ from eqn.(20), it is trivial to show that $J_{trans} = 0$.

3. Relativistic Emission Model in Presence of Vector Potential Barrier (Zeroth Order Approximation)

Now in presence of a triangular shape vector type potential barrier at the interface, we have the conventional form of Dirac equation

$$(\vec{\alpha}.\vec{p} + \beta m)\psi = (E - C + Fx)\psi \tag{22}$$

Decomposing the spinor ψ into u and v as before and using the same representation for $\vec{\alpha}$ and β as mentioned in the previous section, we have

$$\frac{du}{dx} + mu = (E - C + Fx)v \tag{23}$$

$$-\frac{dv}{dx} + mv = (E - C + Fx)u \tag{24}$$

Eliminating v from these two equations, we have

$$-\frac{d^2u}{dx^2} + F\left[\frac{du/dx + mu}{E - C + Fx}\right] + m^2 u = (E - C + Fx)^2 u \tag{25}$$

Substituting $\xi = -(C - E)/F + x$, we have

$$\xi^2 \frac{d^2u}{d\xi^2} - \xi\frac{du}{d\xi} - (m\xi + m^2\xi^2 - F^2\xi^4)u = 0 \tag{26}$$

To obtain an analytical solution for u, we first adopt the non-linear transformation $\eta = F\xi^2/2$. Then the above equation becomes

$$\frac{d^2u}{d\eta^2} + u - \frac{1}{4\eta^2}\left(m\left(\frac{2}{F}\right)^{1/2}\eta^{1/2} + \frac{2\eta m^2}{F}\right)u = 0 \tag{27}$$

In the next step, to obtain a closed analytical solution, let us first neglect for the sake of simplicity the first term within the bracket, i.e., we put $m(2/F)^{1/2}\eta^{1/2} = 0$. After getting the approximate solution from here we substitute it back in the neglected term in eqn.(27) and obtain a more exact result by iterative technique. Using the zeroth order approximation the simplified form of eqn.(27) is given by

$$\frac{d^2u_0}{d\eta^2} + \left(1 - \frac{2\chi}{\eta}\right) u_0 = 0 \tag{28}$$

where $\chi = m^2/4F$ and $\eta > 0$. This is the well-known differential equation for Coulomb function with zero angular momentum[15]. Following[1], we write $C = \mu + W_f$, where W_f is the work function for electron emission from the crustal matter. In some previous work we have shown from the idea of cold cathode emission that in presence of strong quantizing magnetic field, the work function associated with cold electron emission becomes anisotropic[16]. The longitudinal part which is particularly relevant for the present work is given by $W_f = W_c \times (B/B_c^{(e)})^{1/2}$ in eV, where $W_c \approx 82.93$ and $B_c^{(e)} \approx 4.43 \times 10^{13}$G, whereas transverse part is found to be extremely large, in the extreme case, it is infinity in presence of ultra-strong magnetic field. Now in the atomic scale, the quantity F, in the unit of eV/Å is observed to be extremely small, the maximum value is $\sim 10^{-3}$eV/Å. We may therefore safely assume that the newly defined coordinate variable η is large enough, ideally it is $\sim \infty$. The solution of eqn.(28) is then given by the asymptotic form of Coulomb function[15] (see also[17]), given by

$$u_0(\eta) = iC_0[\exp(-i\zeta) - \exp(2i\delta_0)\exp(i\zeta)] = u_I \quad \text{(say)} \tag{29}$$

where $\zeta = \eta - \chi \ln 2\eta + \delta_0$, $\delta_0 = \arg\Gamma(1 + i\chi)$ and C_0 is normalization constant.

To obtain transmission co-efficient, we again consider the continuity of u and its derivative at the interface (i.e., at $x = 0$), which gives

$$u_{II}(0) = u_I(0) \quad \text{and} \quad u'_{II}(0) = u'_I(0) \tag{30}$$

where $u_{II}(x)$ is given by eqn.(14). Hence we get

$$(a + a') = iX_0 \tag{31}$$

where

$$X_0 = C_0 w_k^{1/2}[\exp(-i\zeta_0) - \exp(2i\delta_0)\exp(i\zeta_0)] \tag{32}$$

and

$$(a - a') = -i\frac{C_0}{w_k^{1/2}}\left(1 - \frac{\chi}{\eta}\right)[\exp(-i\zeta_0) + \exp(2i\delta_0)\exp(i\zeta_0)] \tag{33}$$

Where, as before $\eta_0 = \eta(x = 0)$ and $\zeta_0 = \zeta(x = 0)$. Hence, from eqn.(19) and following Appendix-A, we have the transmission coefficient

$$\begin{aligned} T_R &= \frac{4\alpha\beta}{|a|^2}[-\sin^2\zeta_0 + \sin^2(\zeta_0 + 2\delta_0) \\ &- \cos^2\zeta_0 + \cos^2(\zeta_0 + 2\delta_0)] = 0 \end{aligned} \tag{34}$$

Obviously, in this case also like the scalar type surface barrier, there will be no electron field emission.

At the end of Appendix-A we have shown explicitly that for the vector type surface barrier with zeroth order approximation, the transmission current also becomes exactly zero.

4. Relativistic Emission Model in Presence of Vector Potential Barrier (First Order Approximation)

In the previous section, we have noticed that with zeroth order approximate solution, electron field emission is totally forbidden. Let us now consider the first order approximation. To get the next order exact result, we shall substitute the zeroth order solution as given by eqn.(29) into the neglected term on the right hand side of eqn.(27). Then we have

$$(D^2 + \gamma^2)u_1 = \psi \frac{u_0}{\eta^{3/2}} \tag{35}$$

where

$$D = \frac{d}{d\eta}, \quad \gamma = \left(1 - \frac{2\chi}{\eta}\right)^{1/2}, \quad \psi = \frac{m}{4}\left(\frac{2}{F}\right)^{1/2},$$

with u_0 and u_1 are the zeroth and first order solutions respectively. Then as shown in Appendix-B, the first order iterative solution is given by

$$\begin{aligned} u(\eta, \chi) &= iC_0(1 + A + C + iB)\exp(-i\zeta) \\ + &\quad iC_0(-1 - A - C + iB)\exp(2i\delta_0)\exp(i\zeta) \end{aligned} \tag{36}$$

where A, B and C are defined in Appendix-B and also we have shown explicitly in this appendix that with the first order iterative correction, the transmission co-efficient vanishes identically. It is now a matter of simple algebra to show that the transmission co-efficient will remain zero for any higher order iterative correction.

At the end of Appendix-B we have also calculated explicitly the transmission current with first order approximate spinor solution, and similar to the previous two cases, it is also identically zero.

5. Conclusion

In this article we have studied field emission transmission co-efficient for high energy electrons in $1 + 1$-dimension from the polar region of strongly magnetized neutron stars. The potential introduced in Dirac equation following standard relativistic hadro-dynamic technique. Following Fowler and Nordheim, we have considered a triangular type potential barrier at the poles, which is a combination of surface barrier and the attractive potential produced by the magnetic field of the rotating neutron star. From this theoretical investigation we have noticed that the transmission co-efficient of field electron emission vanishes for both scalar type as well as vector type electro-static potential barriers. As an alternative to this, we have also obtained transmission currents for all the cases and are observed to be exactly zero.

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6. Appendix-A

6.1 Transmission Coefficient-I: Zeroth Order Approximation

Following eqns.(31) and (33), we have

$$\begin{aligned} a &= \frac{1}{2}\big[\sin\zeta_0(\alpha-\beta)+\sin(\zeta_0+2\delta_0)(\alpha+\beta) \\ &+ i\{\cos\zeta_0(\alpha-\beta)-\cos(\zeta_0+2\delta_0)(\alpha+\beta)\}\big] \end{aligned} \tag{37}$$

and

$$\begin{aligned} a' &= \frac{1}{2}\big[\sin\zeta_0(\alpha+\beta)+\sin(\zeta_0+2\delta_0)(\alpha-\beta) \\ &+ i\{\cos\zeta_0(\alpha+\beta)-\cos(\zeta_0+2\delta_0)(\alpha-\beta)\}\big] \end{aligned} \tag{38}$$

Where

$$\alpha = C_0 w_k^{1/2} \ \text{ and } \ \beta = \frac{C_0}{w_k^{1/2}}\left(1-\frac{\chi}{\eta}\right)$$

Then from the definition of transmission coefficient, we have

$$T_R = 1 - \frac{|a'|^2}{|a|^2}$$

which can very easily be shown to be exactly zero by simple algebraic manipulation.

6.2 Transmission Current for Zeroth Order Case

In the zeroth order approximation, eqn.(29) may be written as

$$u_0 = A_1 + iB_1 \tag{39}$$

where

$$\begin{aligned} A_1 &= |C_0|[\sin(\zeta - \Delta) + \sin(\zeta + \Delta + 2\delta_0)] \\ \text{and} \quad B_1 &= -|C_0|[\cos(\zeta - \Delta) + cos(\zeta + \Delta + 2\delta_0)] \end{aligned} \tag{40}$$

with

$$C_0 = |C_0| \exp(i\Delta). \tag{41}$$

From eqns.(23) and (24), we have

$$v_0 = C_1 + iD_1 \tag{42}$$

with

$$C_1 = |C_0|A_0[\cos(\zeta - \Delta - \theta) + \cos(\zeta + \Delta - \theta + 2\delta_0)] \tag{43}$$

and

$$D_1 = |C_0|A_0[-\sin(\zeta - \Delta - \theta) + \sin(\zeta + \Delta - \theta + 2\delta_0)] \tag{44}$$

where

$$A_0 \cos\theta = 1 - \frac{\xi}{\eta} \quad \text{and} \quad A_0 \sin\theta = \frac{m}{F\xi} \tag{45}$$

The Dirac spinor is then given by

$$\psi = \begin{pmatrix} A_1 + iB_1 \\ C_1 + iD_1 \end{pmatrix} \tag{46}$$

Then from the definition of transmission current (eqn.(21)), we have

$$J_{trans} = i(A_1 - iB_1 \; C_1 - iD_1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_1 + iB_1 \\ C_1 + iD_1 \end{pmatrix}, \tag{47}$$

Hence

$$J_{trans} = 2(A_1 D_1 - B_1 C_1) \tag{48}$$

Now using eqns.(40)-(45), it is a matter of simple algebra to show that the transmission current becomes exactly zero in the zeroth order approximation.

7. Appendix-B

7.1 Transmission Coefficient-II: First Order Approximation

The solution as given in eqn.(36), which is the particular integral of eqn.(35) and also the first order correction to eqn.(29) can be obtained from

$$\begin{aligned} u_1(\eta, \chi) &= \psi \frac{1}{D^2 + \gamma^2} \frac{u_0}{\eta^{3/2}} \\ &= \frac{\psi}{\eta^{3/2}} \frac{1}{D^2 + \gamma^2} u_0 + \psi u_0 \frac{1}{D^2 + \gamma^2} \frac{1}{\eta^{3/2}} \end{aligned} \tag{49}$$

Using ζ and D as defined after eqn.(29) and eqn.(35) respectively, we have

$$D = \left(1 - \frac{\chi}{\eta}\right) D_\zeta$$

and

$$D^2 = \frac{\chi}{\eta^2} D_\zeta + \left(1 - \frac{\chi}{\eta}\right)^2 D_\zeta^2$$

where $D_\zeta = d/d\zeta$. Then the first term in eqn.(49) is given by

$$\begin{aligned} &= \frac{\psi}{\eta^{3/2}} iC_0 \frac{1}{\left(1 - \frac{\chi}{\eta}\right)^2 D_\zeta^2 + \frac{\chi}{\eta^2} D_\zeta + \gamma^2} \times \\ &\quad (\exp(-i\zeta) - \exp[i(2\delta_0 + \zeta)]) \\ &= iC_0[A(\exp(-i\zeta) - \exp[i(2\delta_0 + \zeta)]) \\ &+ iB(\exp(-i\zeta) + \exp[i(2\delta_0 + \zeta)])] \end{aligned} \tag{50}$$

where

$$A = \frac{\psi}{\eta^{3/2}} \frac{\gamma^2 - \left(1 - \frac{\chi}{\eta}\right)^2}{\left\{\gamma^2 - \left(1 - \frac{\chi}{\eta}\right)^2\right\}^2 + \frac{\chi^2}{\eta^4}} \tag{51}$$

and

$$B = \frac{\psi}{\eta^{3/2}} \frac{\frac{\chi}{\eta^2}}{\left\{\gamma^2 - \left(1 - \frac{\chi}{\eta}\right)^2\right\}^2 + \frac{\chi^2}{\eta^4}} \tag{52}$$

Defining $X = \eta^{-3/2}$, we have

$$D = -\frac{3}{2}\eta^{-5/2} D_X \quad \text{and} \quad D^2 = p^2 D_X^2 + q^2 D_X$$

where

$$D_X = \frac{d}{dX}, p^2 = \frac{15}{4}\eta^{-7/2} \quad \text{and} \quad q^2 = \frac{9}{4}\eta^{-5}$$

Hence the second term reduces to

$$\frac{\psi}{\eta^{3/2}} \frac{1}{q^2 + \gamma^2} u_0 \tag{53}$$

Combining the first and the second term as given by eqns.(50) and (53) respectively and adding with the zeroth order result, given by eqn.(29) and finally properly rearranging, we have the final solution

$$\begin{aligned} u(\eta, \chi) &= iC_0[(1 + A + C + iB)\exp(-i\zeta) \\ &+ (-1 - A - C + iB)\exp(2i\delta_0)\exp(i\zeta)] \end{aligned} \tag{54}$$

Using the boundary conditions as used before at $x = 0$, we have

$$\begin{aligned} a + a' &= w_k^{1/2}[(iX_1 - X_2)\exp(-i\zeta_0) - (iX_1 + X_2) \times \\ &\quad \exp\{i(2\delta_0 + \zeta_0)\}] \end{aligned} \tag{55}$$

and

$$
\begin{aligned}
a - a' &= \frac{1}{w_k^{1/2}}\left(1-\frac{\chi}{\eta}\right)[(X_2 - iX_1)\exp(-i\zeta_0) - (iX_1 + X_2)\times \\
&\quad \exp\{i(2\delta_0+\zeta_0)\}] + \frac{1}{w_k^{1/2}}[(iX_1' - X_2')\exp(-i\zeta_0) \\
&- (iX_1' + X_2')\exp\{i(2\delta_0+\zeta_0)\}]
\end{aligned}
\tag{56}
$$

Hence one can show very easily that the transmission coefficient vanishes exactly, i.e.,

$$T_R = 1 - \frac{\mid a' \mid^2}{\mid a \mid^2} = 0 \tag{57}$$

7.2 Transmission Current for First Order Approximation

We can express eqn.(36) in a more convenient form, given by

$$u = iC_0[(p_1 + ip_2)\exp(-i\zeta) - (p_1 - ip_2)\exp\{i(\zeta + 2\delta_0)\}] \tag{58}$$

where $p_1 = 1 + A + C$ and $p_2 = B$ and are functions of x or equivalently ξ, or η or ζ. Then as before the above equation may also be written in the form

$$u = A_1 + iB_1 \tag{59}$$

Using the relation $E - C + Fx = F\xi$ as defined after eqn.(25). we have

$$\frac{1}{F\xi}mu = i\frac{mC_0}{F\xi}[(p_1 + ip_2)\exp(-i\zeta) - (p_1 - ip_2)\exp\{i(\zeta + 2\delta_0)\}] \tag{60}$$

While taking derivatives of u, we have to consider the exponential terms containing ζ and also p_1, p_2, which are also functions of ζ. Let us first consider the derivatives of exponential terms. We indicate it by $u'_{(1)}$. Then the corresponding term from eqn.(23) is given by

$$\frac{1}{F\xi}u'_{(1)} = \frac{C_0}{F\xi}\left[F\xi\left(1-\frac{\chi}{\eta}\right)\{(p_1 + ip_2)\exp(-i\zeta) + (p_1 - ip_2)\exp\{i(\zeta + 2\delta_0)\}\}\right] \tag{61}$$

We next consider the term containing the derivatives of p_1 and p_2 with respect to ζ, given by

$$\frac{1}{F\xi}u'_{(2)} = i\frac{C_0}{F\xi}[(p_1' + ip_2')\exp(-i\zeta) - (p_1' - ip_2')\exp\{i(\zeta + 2\delta_0)\}] \tag{62}$$

Combining eqns.(60), (61) and (62), we finally have

$$v = \mid C_0 \mid A_0\,[\exp\{-i(\zeta - \Delta - \theta)\} + \exp\{i(\zeta + \Delta - \theta + 2\delta_0)\}] \tag{63}$$

where we have used eqn.(41) and defined

$$A_0\cos\theta = \alpha \quad \text{and} \quad A_0\sin\theta = \beta \tag{64}$$

with

$$\alpha = -mp_2 + F\xi\left(1-\frac{\chi}{\eta}\right)p_1 - p_2' \quad \text{and} \quad \beta = mp_1 + F\xi\left(1-\frac{\chi}{\eta}\right)p_2 + p_1' \tag{65}$$

Hence writing

$$v = C_1 + iD_1 \tag{66}$$

where

$$C_1 = \mid C_0 \mid A_0[\cos(\zeta - \Delta - \theta) + \cos(\zeta + \Delta - \theta + 2\delta_0)] \tag{67}$$

and

$$D_1 = \mid C_0 \mid A_0[-\sin(\zeta - \Delta - \theta) + \sin(\zeta + \Delta - \theta + 2\delta_0)] \tag{68}$$

it is trivial to show from eqn.(48) that the transmission current in this case also vanishes explicitly.