The Aspects of Rotating Frame – An Electromagnetic Analogy

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Abstract

In this article, various non-relativistic motions of a body have been studied from the viewpoints of both an ***inertial*** and a ***rotating*** **(*non-inertial)*** *observer*, thus illustrating the major differences in the *ways of description* between those two viewpoints. As an application, we have illustrated classical description of ***Larmor precession***. Furthermore, it has been shown that description of motion of a body (***mass***) given by a *rotating observer* has vast similarity with description of motion of a ***charged particle*** moving in an ***electromagnetic field***, given by an *inertial observer*. Validity and implication of ***work-energy theorem*** in a rotating frame, different work energy considerations and their side-by-side correspondence in rotating and electromagnetic cases have been discussed. Furthermore, similarity between ***velocity dependent potential*** associated with the '*pseudo' forces* in a *rotating* frame and with *electromagnetic interaction forces* has been studied, illustrating the concept of ***effective potential*** in course of rotating frame description of motion of a body. Also, a set of equations similar to ***Maxwell's equations*** *in electrodynamics* has been designed in case of rotating frame description of motion, for some simplified assumptions about variation of angular velocity of a rotating frame.

1. Introduction: The Two Frames and the Transformation Equation

Newton's laws of motion are applicable in any *inertial frame* (which is defined by *Newton's 1st law* itself), but not in any non-inertial frame, which accelerates with respect to an inertial frame. Hence, a non-inertial observer cannot describe motion of a body by applying Newton's laws directly. He can't just put: $\mathbf{F}_{\text{ext}} = m\mathbf{a}$ (where $\mathbf{a}$ is acceleration of the body seen by the non-inertial observer and $\mathbf{F}_{\text{ext}}$ is the external force, which is due to some kind of interaction) using *Newton's 2nd law*. But the non-inertial observer can indeed describe the motion of the body, if he takes into account the *'pseudo'* (or *inertial*) forces (along with the external forces), which come into the scene due to *relative acceleration between the frames* (and not due to any physical interaction).

Total time derivative i.e. time rate of change of any dynamical variable vector as observed by an inertial observer and a non-inertial observer (rotating with an angular velocity $\boldsymbol{\omega}$ with respect to the inertial one) are related by an *operator relationship*[1]:

$$\left(\frac{d}{dt}\right)_{\mathrm{i}} \equiv \left(\frac{d}{dt}\right)_{\mathrm{r}} + \boldsymbol{\omega}\times \qquad \dots\dots\dots(1)$$

Here, and elsewhere in this text, subscripts 'i' and 'r' denotes the corresponding quantity as observed by the inertial and the rotating observers respectively. Using the operator relationship-(1), we get a *transformation equation*[1] relating forces and accelerations of a body of mass *m* observed by the two observers. The transformation equation is as follows:

$$m\mathbf{a}_{\mathrm{r}} = \mathbf{F}_{\mathrm{ext}} + \mathbf{F}_{\mathrm{pseudo}} \qquad \dots\dots\dots(2)$$

where, $$\mathbf{F}_{\mathrm{pseudo}} = -2m\boldsymbol{\omega}\times\mathbf{v}_{\mathrm{r}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r} \qquad \dots\dots\dots(3)$$

Here, $\mathbf{a}_{\mathrm{r}}$: Acceleration of the body as observed by the rotating observer,

$\mathbf{F}_{\mathrm{ext}}$: External (interaction) force (Note that in the Inertial frame, $\mathbf{F}_{\mathrm{ext}} = m\mathbf{a}$ where, $\mathbf{a}$ is the acceleration observed by the inertial observer)

$\mathbf{F}_{\mathrm{pseudo}}$: Resultant 'pseudo' force (arising due to relative acceleration of the frames) which comprises of the following pseudo force terms:

- $\mathbf{F}_{\mathrm{centrifugal}} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$: Centrifugal force,
- $\mathbf{F}_{\mathrm{Coriolis}} = -2m\boldsymbol{\omega}\times\mathbf{v}_{\mathrm{r}}$: Coriolis force,
- $\mathbf{F}_{\mathrm{Euler}} = -m\dot{\boldsymbol{\omega}}\times\mathbf{r}$: Euler force.

where, $\mathbf{r}$ is the position vector of the body w.r.t. the origin (centre of the disc), and $\mathbf{v}_{\mathrm{r}}$ is the velocity of the body *observed by the rotating observer*.

Note that the pseudo forces are felt *only by the non-inertial observer*, who feels some kind of 'forcing effect', but can't find any interactive source of its arousal, and hence cannot identify it as an interaction force.

2. Two Viewpoints

Now, suppose we have a predefined *inertial* frame and a horizontal disc rotating with *uniform* angular velocity $\boldsymbol{\omega}$ (perpendicular to the plane of the disc) with respect to the inertial frame, i.e., a *rotating* (non-inertial) frame. Suppose *a man* (our observable) is moving ***on the rotating disc***. We assume that there is always sufficient friction between the man and the disc to balance the man.

Here, the problem has *cylindrical symmetry*, and we can treat general three dimensional motion of a body by considering the *radial*, *circumferential* (or tangential to a circle concentric with the disc) and *vertical* components of the kinematic variable vectors (e.g. position, velocity etc.) separately. Hence we shall illustrate description of radial, circumferential and vertical motions only. Afterwards, we shall discuss another case, where the man will be moving with constant velocity in the inertial frame (say, **on the ground, *outside the disc***), and describe it from the viewpoint of the rotating observer. So, let's first discuss three cases of motion of the man on the rotating disc, and find how the two observers describe his motion.

- **CASE-1:** ***Man is moving ON THE DISC, towards the centre of the disc radially inward, with a constant velocity*** $\mathbf{v}_\mathrm{r}$ ***with respect to the disc:***

Rotating Observer: His description is very straight-forward. The acceleration of the man with respect to the rotating observer is zero. Hence, considering static equilibrium of the man under the different forces (*see fig.2a*), from the transformation equation (as the angular velocity of the disc is constant, hence there will be no Euler force term):

$$\mathbf{0} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{coriolis}} + \mathbf{F}_{\text{centrifugal}}$$

$$= \mathbf{F}_{\text{friction}} - m[2\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$$

or, $$\mathbf{F}_{\text{friction}} = m[2\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] \qquad \text{(the equation of motion)}$$

Inertial Observer: The velocity of the man on the disc observed by the outside inertial observer (*see fig.2b*) is given by:

$$\mathbf{v} = \left.\frac{d\mathbf{r}}{dt}\right|_\mathrm{i} = \left.\frac{d\mathbf{r}}{dt}\right|_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r}$$

And hence the acceleration observed by the inertial observer is,

$$\mathbf{a} = \left.\frac{d}{dt}(\mathbf{v})\right|_\mathrm{i} = \left.\frac{d}{dt}(\mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r})\right|_\mathrm{i}$$

$$= \left.\frac{d\mathbf{v}_\mathrm{r}}{dt}\right|_\mathrm{i} + \left.\frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r})\right|_\mathrm{i}$$

$$= \left[\left(\frac{d\mathbf{v}_\mathrm{r}}{dt}\right)_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{v}_\mathrm{r}\right] + \left.\frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r})\right|_\mathrm{i}$$

$$= \mathbf{0} + \boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \left.\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right)\right|_\mathrm{i} + \left.\left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right)\right|_\mathrm{i}$$

$$= \boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \mathbf{0} + \boldsymbol{\omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_\mathrm{r} + \boldsymbol{\omega} \times \mathbf{r}\right]$$

$$= \boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times [\mathbf{v}_\mathrm{r} + (\boldsymbol{\omega} \times \mathbf{r})]$$

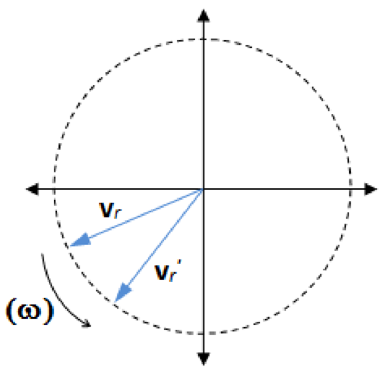

Figure 1: The rotating v_r vector, observed by inertial observer

or, $$\mathbf{F}_{\text{friction}} = m\mathbf{a} = m[2\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$$

Thus, we arrived at the same equation of motion. Note the significance of the terms here (*see fig.1*): contribution in acceleration of the man observed from inertial frame comes from two parts. One is $\left.\frac{d\mathbf{v}_\mathrm{r}}{dt}\right|_\mathrm{i}$ which gives rate of change of velocity on rotating frame as viewed from the inertial frame, and another is $\left.\frac{d}{dt}(\boldsymbol{\omega}\times\mathbf{r})\right|_\mathrm{i}$ which is due to the motion of the rotating frame itself. Further, we can see $(\boldsymbol{\omega}\times\mathbf{v}_\mathrm{r})$ actually gives a *vector* $\mathbf{v}_\mathrm{r}$ *rotating with angular velocity* $\boldsymbol{\omega}$ (*see fig.1*). Whereas, $m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is just mass times *centripetal acceleration* which keeps the man rotating. Summing up, what the inertial observer sees is that, the man is approaching towards the centre in a *two dimensional spiral path* (the direction of $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is radially inward).

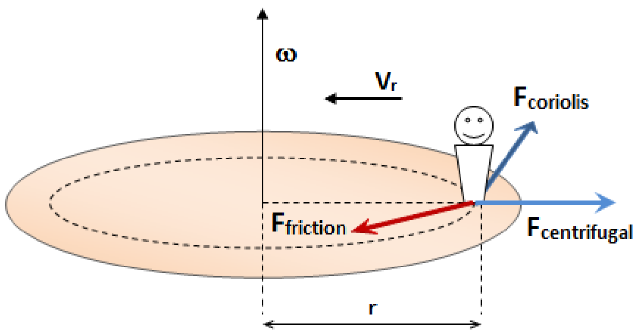

Figure 2a: Description by Rotating Observer Figure 2b: Description by Inertial Observer

Figure 2: Radial motion

- **<u>CASE-2</u>:** ***<u>Man is moving in a tangential (to a circle concentric with the disc) direction with velocity $\mathbf{v}_\mathrm{r}=v_\mathrm{r}\hat{\mathbf{t}}$ on the disc:</u>***

<u>Inertial Observer:</u> To the inertial observer, the velocity of the man on the disc is:

$$\mathbf{v}=\left.\frac{d\mathbf{r}}{dt}\right|_\mathrm{i}=\left.\frac{d\mathbf{r}}{dt}\right|_\mathrm{r}+\boldsymbol{\omega}\times\mathbf{r}=v_\mathrm{r}\hat{\mathbf{t}}+v_\mathrm{t}\hat{\mathbf{t}}=(v_\mathrm{r}+v_\mathrm{t})\hat{\mathbf{t}}$$

[$\hat{\mathbf{t}}$ is a unit vector along tangential direction same as of $\boldsymbol{\omega}\times\mathbf{r}$]

where, $v_\mathrm{t}\hat{\mathbf{t}}\,(=\boldsymbol{\omega}\times\mathbf{r})$ is the linear velocity of the disc at the position of the man.

Hence, to him, it seems that the man (*see fig.3a*) is standing on another disc that is rotating with a uniform angular velocity, $\boldsymbol{\omega}'$, such that,

$$(v_\mathrm{r}+v_\mathrm{t})\hat{\mathbf{t}}=\boldsymbol{\omega}'\times\mathbf{r}=\omega' r\hat{\mathbf{t}}$$

Thus, the equation of motion of the man:

$$\mathbf{F}_\mathrm{friction}=m\omega'^2 r.(-\hat{\mathbf{r}})$$

i.e., $\mathbf{F}_{\text{friction}} = m\frac{v^2}{r}.(-\hat{\mathbf{r}}) = m\frac{(v_r+v_t)^2}{r}.(-\hat{\mathbf{r}})$

or, $\mathbf{F}_{\text{friction}} = \frac{m{v_r}^2}{r}.(-\hat{\mathbf{r}}) + \frac{2mv_tv_r}{r}.(-\hat{\mathbf{r}}) + \frac{m{v_t}^2}{r}.(-\hat{\mathbf{r}})$

<u>Rotating Observer:</u> To the rotating observer, the man on the disc has a centripetal acceleration $\mathbf{a}_r$ with respect to the rotating disc (*see fig.3b*).

Thus, $m\mathbf{a}_r = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{coriolis}} + \mathbf{F}_{\text{centrifugal}}$

or, $m\frac{{v_r}^2}{r}.(-\hat{\mathbf{r}}) = \mathbf{F}_{\text{friction}} - 2m\boldsymbol{\omega} \times \mathbf{v}_r - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

or, $m\frac{{v_r}^2}{r}.(-\hat{\mathbf{r}}) = \mathbf{F}_{\text{friction}} - 2m\frac{v_t}{r}.(\hat{\boldsymbol{\omega}}) \times v_r.(\hat{\mathbf{t}}) - \frac{{v_t}^2}{r}.(-\hat{\mathbf{r}})$

or, $\mathbf{F}_{\text{friction}} = \frac{m{v_r}^2}{r}.(-\hat{\mathbf{r}}) + \frac{2mv_tv_r}{r}.(-\hat{\mathbf{r}}) + \frac{m{v_t}^2}{r}.(-\hat{\mathbf{r}})$

[here, $\mathbf{v}_t = v_t.(\hat{\mathbf{t}}) = \boldsymbol{\omega} \times \mathbf{r} = \omega r.(\hat{\mathbf{t}})$]

Hence, once again we reached the same equation of motion, treating the problem from the two different frames of reference.

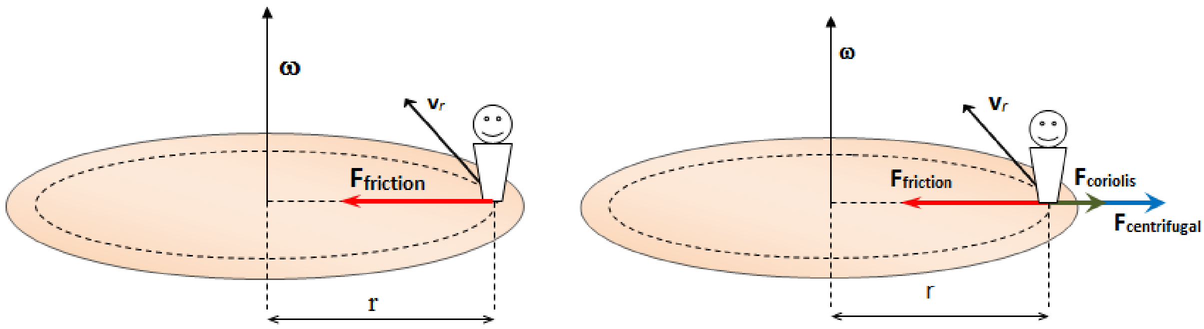

Figure 3a: Inertial observer's Description **Figure 3b: Rotating observer's description**

Figure 3: Tangential Motion

- **CASE-3:** ***<u>Man is moving vertically upward with a constant velocity</u>*** $\mathbf{v}_r$ ***<u>with respect to the rotating frame:</u>***

<u>Rotating Observer:</u> To the rotating observer, the acceleration of the man is zero. Hence, from the transformation equation:

$$\mathbf{0} = \frac{\mathbf{F}_{\text{ext}}}{m} - 2\boldsymbol{\omega} \times \mathbf{v}_r - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

or, $\mathbf{0} = \frac{\mathbf{F}_{\text{ext}}}{m} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

[as $\boldsymbol{\omega}$ and $\mathbf{v}_r$ are parallel]

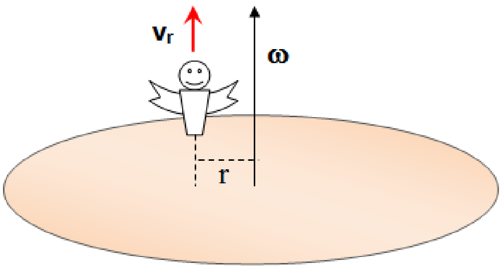

Figure 4: Man moving vertically upward

Inertial Observer: To the inertial observer, the velocity of the man is,

$$\mathbf{v} = \left.\frac{d\mathbf{r}}{dt}\right|_i = \left.\frac{d\mathbf{r}}{dt}\right|_r + \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$$

And hence the acceleration is,

$$\mathbf{a} = \left.\frac{d\mathbf{v}}{dt}\right|_i = \left.\frac{d}{dt}(\mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r})\right|_i = \left.\frac{d\mathbf{v}_r}{dt}\right|_i + \left.\frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r})\right|_i$$

$$= \left[\left(\frac{d\mathbf{v}_r}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{v}_r\right] + \left.\frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r})\right|_i$$

$$= \mathbf{0} + \mathbf{0} + \left.\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right)\right|_i + \left.\left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right)\right|_i$$

$$= \mathbf{0} + \boldsymbol{\omega} \times \left[\left(\frac{d\mathbf{r}}{dt}\right)_r + \boldsymbol{\omega} \times \mathbf{r}\right]$$

$$= \boldsymbol{\omega} \times [\mathbf{v}_r + (\boldsymbol{\omega} \times \mathbf{r})]$$

$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad \text{[Centripetal acceleration]}$$

Hence, $\frac{\mathbf{F}_{\text{ext}}}{m} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$, i.e., once again we got back the same equation.

3. Things Reversed

In this section, we are going to deal with the *reverse* of the scene discussed above. We are going to illustrate only one case here. The other cases can then be done in a similar fashion.

We consider now, a man (our observable) in the *inertial frame (say, ON THE GROUND)* moving away with a constant velocity $\mathbf{v}_i$ from the rotating disc, along the line passing through the centre of the disc, radially outward.

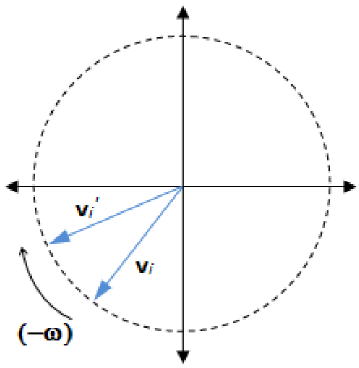

Figure 5: The rotating $\mathbf{v}_i$ vector

Inertial Observer: From the inertial frame, description of the man's motion is very straight forward. The inertial observer will just say that the man is acted upon by no external force, and hence, is moving with a constant velocity $\mathbf{v}_i = \mathbf{v}$, say (*see fig.6a*), as it should be according to Newton's 1st Law.

Rotating Observer: The rotating observer will write down the transformation equation and take the pseudo forces into account (*see fig.6b*), as follows:

$$m\mathbf{a}_{\mathrm{r}} = \mathbf{F}_{\mathrm{ext}} - 2m\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{r}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \mathbf{0} - 2m\boldsymbol{\omega} \times (\mathbf{v}_{\mathrm{i}} - \boldsymbol{\omega} \times \mathbf{r}) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= -2m\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{i}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

We may understand the significance of these two terms as follows (*see fig.5*): The first term just tells that the vector $\mathbf{v}_{\mathrm{i}}$ rotates with angular velocity of *same magnitude* ω *but in a reverse direction*. The second term is *mass times a centripetal acceleration* needed for the rotation of the man around the disc as observed by the rotating observer. Note that, above, in the first line, the last two terms are respectively the *coriolis* and the *centrifugal* force terms. But in the third line, those two terms have *combined and given rise to two different terms*, whose significance are as just said. Hence the rotating observer will see the man to be moving outward in a *two dimensional spiral path*, the spiral being in an *opposite sense* to that of the direction of rotation of the disc.

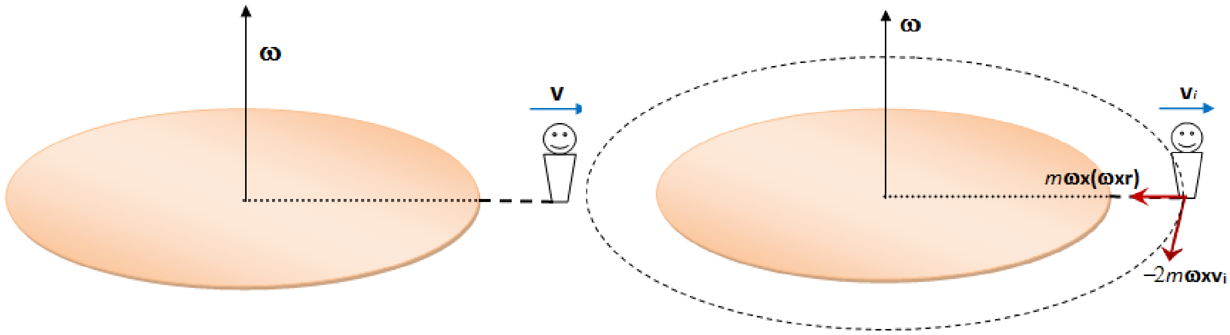

Figure 6a: Observation of inertial observer

Figure 6b: Observation of Rotating Observer

4. Larmor Precession

As an application of the above discussed topics, in this section, the phenomenon of Larmor precession has been illustrated from two frames. We consider a single-electron-atom. We shall not consider the electron's spin, only the orbital motion of the electron will be dealt with. For that purpose, we shall consider the electron to be moving in circular trajectory, under the action of the Coulombic attractive force of the nucleus. We shall assume the nucleus to be a point at rest in the inertial frame (due to its high mass compared to the electron, and negligible spatial dimensions).

In absence of any external fields, orbital angular momentum $\mathbf{L}$ (about the nucleus point) and corresponding orbital magnetic moment $\mathbf{M}$ are related by:

$$\mathbf{M} = -\frac{e}{2m}\mathbf{L}$$

Where $-e$ $(e > 0)$ and m are respectively the charge and the mass of the electron.

Now, if an external magnetic field $\mathbf{B}$ is applied, then along with the Coulombic force the electron is additionally acted upon by the Lorentz force:

$$\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$$

where, $\mathbf{v}$ is velocity of the electron at any point on its trajectory, observed by inertial observer.

Now, according to the inertial observer, the electron is acted upon by an additional magnetic force as above which is always directed *perpendicular* to the direction of its motion, and hence is a *no-work force*. Let's see the effect of the torque generated due to interaction of the atomic magnetic dipole (the rotating electron) with the magnetic field. The torque about the point nucleus is given by:

$$\mathbf{N} = \mathbf{M} \times \mathbf{B} = -\frac{e}{2m}\mathbf{L} \times \mathbf{B} = \boldsymbol{\omega}_{\mathrm{L}} \times \mathbf{L}$$

Where, $$\boldsymbol{\omega}_{\mathrm{L}} \equiv \frac{e}{2m}\mathbf{B}$$

As torque $\mathbf{N} = \frac{d}{dt}\mathbf{L}$, the time rate of change of angular momentum,

$$\frac{d}{dt}\mathbf{L} = \boldsymbol{\omega}_{\mathrm{L}} \times \mathbf{L}$$

By our previous operator relation of equation-(1), we note that when we observe $\left(\frac{d}{dt}\mathbf{L}\right)$ from a rotating frame with centre at the nucleus, rotating with the angular velocity $\boldsymbol{\omega}_{\mathrm{L}}$, it would be $\left(\frac{d}{dt}\mathbf{L}\right)_{\mathrm{r}} = \mathbf{0}$, implying that the vector $\mathbf{L}$ is constant in that frame. Hence we can say, that w.r.t. the inertial observer, $\mathbf{L}$ precesses about the direction of $\boldsymbol{\omega}_{\mathrm{L}}$ (or equivalently direction of $\mathbf{B}$) with an angular velocity $\boldsymbol{\omega}_{\mathrm{L}}$ [Note that the equation $\frac{d}{dt}\mathbf{L} = \boldsymbol{\omega}_{\mathrm{L}} \times \mathbf{L}$ automatically says that $|\mathbf{L}| = L =$ constant, because, $\left(\frac{d}{dt}\mathbf{L}\right).\mathbf{L} = \boldsymbol{\omega}_{\mathrm{L}} \times \mathbf{L}.\mathbf{L} \Rightarrow \frac{d}{dt}(L^2) = 0 \Rightarrow L =$ constant.]

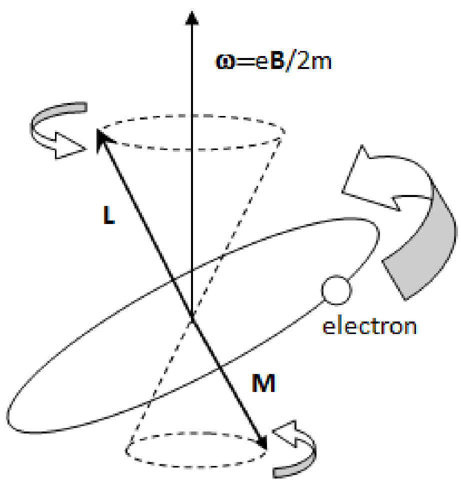

Figure 7: Larmor Precession ($\omega = \omega_{\mathrm{L}}$)

So, what we observe from the inertial viewpoint can be summarised as follows: As the magnetic force $\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$ is always perpendicular to $\mathbf{v}$, and initially $\mathbf{v}$ was directed tangential to the circular path, the magnetic force cannot cause radial displacement of the electron. But the resulting torque causes precession of the vector $\mathbf{L}$ as said. Moreover, the original circular motion of the electron was due to Coulombic attractive force of the nucleus, which is a central force and depends only on the radial distance of the electron from the nucleus (other than the charges of electron and nucleus). Hence, the effect of Coulombic force remains same always, even if $\mathbf{L}$

precesses in aforesaid manner. Hence, in effect, this is previous motion (before applying the magnetic field), plus a precession of the orbit itself. Here, clearly, the Coulombic force (perpendicular to direction of motion), plus Lorentz force (also perpendicular to direction of motion) gives necessary centripetal acceleration required for the electron to move in the ever-changing circular orbit (orientation of the orbit keeps changing due to precession of **L**).

Now, let's try to see this whole picture from a rotating frame rotating with angular velocity $\boldsymbol{\omega}_\mathrm{L}$ about the nucleus. To the first approximation, the motion of the electron should be same as that observed by the inertial observer in absence of the field $\mathbf{B}$. The forces which the rotating observer finds are the Coulombic attraction force, the Lorentz force and pseudo forces. Let's see what the pseudo force terms give us. The Coriolis force term is:

$$\mathbf{F}_\mathrm{Cor} = -2m\,\boldsymbol{\omega}_\mathrm{L} \times \mathbf{v}_\mathrm{r} \qquad (\mathbf{v}_\mathrm{r} : \text{velocity of electron observed from the rotating frame})$$

$$= -2m\frac{e}{2m}\mathbf{B} \times (\mathbf{v} - \boldsymbol{\omega}_\mathrm{L} \times \mathbf{r})$$

$$= e\mathbf{v} \times \mathbf{B} + 2m\,\boldsymbol{\omega}_\mathrm{L} \times (\boldsymbol{\omega}_\mathrm{L} \times \mathbf{r})$$

Note that the first part, namely $(e\mathbf{v} \times \mathbf{B})$ is equal and opposite to the Lorentz force, and in effect, cancels the effect of the Lorentz force. The second term $\{2m\,\boldsymbol{\omega}_\mathrm{L} \times (\boldsymbol{\omega}_\mathrm{L} \times \mathbf{r})\}$ when added with the centrifugal force term $\{-m\boldsymbol{\omega}_\mathrm{L} \times (\boldsymbol{\omega}_\mathrm{L} \times \mathbf{r})\}$, gives $\{m\,\boldsymbol{\omega}_\mathrm{L} \times (\boldsymbol{\omega}_\mathrm{L} \times \mathbf{r})\}$. Now, note that this term is proportional to the radial distance of the electron from the nucleus (very small), and on ${\omega_L}^2$,i.e., on B^2. In a first approximation, when quantities proportional to only the first power of the field B are considered, the $\{m\,\boldsymbol{\omega}_\mathrm{L} \times (\boldsymbol{\omega}_\mathrm{L} \times \mathbf{r})\}$ part becomes negligible, and hence need not be considered. Thus, the electron is under the influence of the Coulombic force only, and from the initial motion of the electron, we may conclude that from the rotating frame, it seems to move in a circular trajectory, as earlier observed by the inertial observer, in absence of the magnetic field.

5. Electromagnetism – Velocity Dependent Potential

We know that another way of writing the equation of motion of a particle is through Lagrangian formulation. On the way of reaching Lagrange's equation of motion, we get an equation[2]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_\mathrm{j}}\right) - \frac{\partial T}{\partial x_\mathrm{j}} = F_\mathrm{j} \qquad \dots\dots\dots(4)$$

where, T is kinetic energy of system in consideration, x_j is jth generalised coordinate and F_j is jth component of generalised force.

Now, in case of the motion of a charged particle in an electromagnetic field, the force can be expressed in terms of a velocity dependent potential:

$$F_{\mathrm{j}} = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_{\mathrm{j}}}\right) - \frac{\partial U}{\partial x_{\mathrm{j}}} \qquad \dots\dots\dots(5)$$

where $U \equiv U(x, \dot{x}, t)$ is the *velocity dependent potential* (the subscript-less x and $\dot{x}$ represent collection of all corresponding indexed variables).

Then from equation-(4), we can immediately reach *Lagrange's equation*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_{\mathrm{j}}}\right) - \frac{\partial L}{\partial x_{\mathrm{j}}} = 0 \qquad \dots\dots\dots(6)$$

where, $L \equiv T - U$ is the *Lagrangian* of the system.

We can show how U makes its entry into the scene, where a charged particle is moving in an electromagnetic field. From Maxwell's equation $\mathbf{\nabla}.\mathbf{B} = 0$, we can write: $\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$, where **A** is a vector function of space coordinates and time, called the *vector potential*. Substituting this expression in Maxwell's another equation $\mathbf{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$, we can write: $\mathbf{E} = -\mathbf{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial t}$, where φ is a scalar function of space coordinates and time, called the *scalar potential*. Thus we can write the expression of *Lorentz force* on a charged particle carrying charge q and moving with velocity **v** due to presence of electromagnetic field as:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$= q\left\{-\mathbf{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\mathbf{\nabla} \times \mathbf{A})\right\} \qquad \dots\dots\dots(7)$$

$$\text{or,} \quad F_{\mathrm{j}} = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_{\mathrm{j}}}\right) - \frac{\partial U}{\partial x_{\mathrm{j}}}$$

where, $U \equiv q\varphi - q\mathbf{A}.\mathbf{v}$ $\dots\dots\dots(8)$ is the *velocity dependent potential associated with the electromagnetic field* we were talking about. The corresponding Lagrangian is: $L = T - U = \frac{1}{2}mv^2 - q\varphi + q\mathbf{A}.\mathbf{v}$ $\dots\dots\dots(9)$

6. Analogy Between Rotating Frame Description of Motion of a Body and Inertial Frame description of Motion of a Charged Particle in an Electromagnetic Field

- **Similarity regarding potential and Lagrangian:**

We have seen, when we are in a rotating frame of reference (rotating with angular velocity $\boldsymbol{\omega}$), we are, at all times, acted upon by three new 'forces' which we did not have in an inertial frame, namely *centrifugal force*, *Coriolis force* and *Euler force*. An interesting feature of these pseudo forces arises if we write them in a slightly rearranged way, as will be shown below. We shall see that just like the electromagnetic case, we can associate with the *resultant pseudo force*, a *velocity dependent potential*. Let's see what all these mean. Slightly rearranging, we can rewrite equation-(3) as:

$$\mathbf{F}_{\text{pseudo}} = -m\boldsymbol{\nabla}\left(-\frac{1}{2}|\boldsymbol{\omega}\times\mathbf{r}|^2\right) - m\frac{\partial}{\partial t}(\boldsymbol{\omega}\times\mathbf{r}) + m\mathbf{v}_{\text{r}}\times\{\boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\mathbf{r})\} \qquad \dots\dots\dots(10)$$

Now comparing equation-(10) with equation-(7), we see, they are of exactly the same form, provided we replace *charge q by mass m*, *velocity* **v** (*with respect to inertial frame*) by $\mathbf{v}_{\text{r}}$ (*with respect to rotating frame*) and define:

$$\mathbf{A_r} \equiv \boldsymbol{\omega}\times \text{r} \qquad \dots\dots\dots(11.a)$$

and $$\varphi_{\text{r}} \equiv -\frac{1}{2}|\boldsymbol{\omega}\times\mathbf{r}|^2 = -\frac{{A_r}^2}{2} \qquad \dots\dots\dots(11.b)$$

Hence the velocity dependent potential associated with the resultant pseudo force in a rotating frame is given by:

$$U_r \equiv m\varphi_{\text{r}} - m\mathbf{A_r}.\mathbf{v}_{\text{r}}$$

$$= -\frac{1}{2}m|\boldsymbol{\omega}\times\mathbf{r}|^2 - m\boldsymbol{\omega}\times\mathbf{r}.\mathbf{v}_{\text{r}} \qquad \dots\dots\dots(12)$$

Now, if we assume that the external force is also conservative, i.e. $\mathbf{F} = -\boldsymbol{\nabla}V$,

Then the total or effective potential is: $V_{\text{eff}} = U_r + V$,

Now, if we try to write a Lagrangian in the rotating frame, that would be something like:

$$L_{\text{r}} = \frac{1}{2}m{v_{\text{r}}}^2 - V_{\text{eff}} \qquad \dots\dots\dots(13)$$

To check that this formulation is correct, we substitute the transformation relations into equation-(13) (we should then get back our old Lagrangian). This gives,

$$L_{\text{r}} = \frac{1}{2}m|\mathbf{v}-\boldsymbol{\omega}\times\mathbf{r}|^2 - (V + U_r)$$

$$= \left(\frac{1}{2}mv^2 - V\right) - m\mathbf{v}.\boldsymbol{\omega}\times\mathbf{r} + \frac{1}{2}m|\boldsymbol{\omega}\times\mathbf{r}|^2 + \frac{1}{2}m|\boldsymbol{\omega}\times\mathbf{r}|^2 + m\boldsymbol{\omega}\times\mathbf{r}.\mathbf{v}_{\text{r}}$$

$$= L \qquad \text{(}\mathbf{v}\text{ is velocity in inertial frame)}$$

which is the Lagrangian in the inertial frame. This indicates that the formulation in equation-(13) is indeed correct.

- ***NOTE:*** Note an important distinction between the potentials in two cases. Whereas in electromagnetic case, the scalar potential and the components of the vector potential were 4 independent scalars, in the rotational case, they are not. Rather, the scalar potential is actually negative half of square of magnitude of the vector potential (thus giving 3 independent scalars). This is due to the fact that, whereas electromagnetic fields and potentials transform under Lorentz transformation, in case of rotating frame such thing is not true. Because, here we take time *t* to be absolute.

- **Similarity regarding fields and forces:**

In magnetic case, the scalar and vector potentials are φ and $\mathbf{A}$ respectively. The corresponding electric and magnetic fields are obtained by respectively:

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial\, t} \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

Let's calculate the analogues of electric and magnetic fields in the mechanical case.

Analogue of Magnetic Force:

From equation-(11.a), the *analogue of magnetic field* is:

$$\mathbf{B_r} = \boldsymbol{\nabla} \times\ \mathbf{A_r} = \boldsymbol{\nabla} \times\ (\boldsymbol{\omega} \times \mathbf{r}) = 2\boldsymbol{\omega}$$

Hence, analogue of magnetic force is:

$$m\mathbf{v}_\mathrm{r} \times\ (\boldsymbol{\nabla} \times\ \mathbf{A_r}) =\ m\mathbf{v}_\mathrm{r} \times 2\boldsymbol{\omega} = -2m\boldsymbol{\omega} \times \mathbf{v}_\mathrm{r}$$

which is just the ***Coriolis*** force (dependent on particle's velocity in the rotating frame).

Analogue of Electric Force:

From equation-(11.b), the *analogue of electric field* is:

$$\mathbf{E_r} = -\boldsymbol{\nabla}\varphi_\mathrm{r} - \frac{\partial \mathbf{A_r}}{\partial\, t} = \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\omega} \times \mathbf{r}|^2\right) - \frac{\partial}{\partial\, t}(\boldsymbol{\omega} \times \mathbf{r}) = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \dot{\boldsymbol{\omega}} \times \mathbf{r}$$

Hence the analogue of force due to electric field is:

$$m\left[\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\omega} \times \mathbf{r}|^2\right) - \frac{\partial}{\partial\, t}(\boldsymbol{\omega} \times \mathrm{r})\right]$$

which is the part of the resultant pseudo force, excluding the Coriolis term, i.e. this part contains the ***centrifugal*** and the ***Euler*** forces (which does not depend on the particle's velocity in the rotating frame). Furthermore, the quantity $\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\omega} \times \mathbf{r}|^2\right)$ is ***conservative*** and hence is analogous to the *conservative electric field* (which arises due to static charge distribution) and $\frac{\partial}{\partial\, t}(\boldsymbol{\omega} \times \mathrm{r})$ is ***non-conservative*** and hence is analogous to the *non-conservative induced electric field* (which arises due to time varying magnetic field).

What we found above is that being in a rotating frame means everything we observe, is in an *extra potential* U_r (associated with the pseudo forces). If we take this fact into account properly, we can as well describe motion of any particle in a similar manner as followed in Newton's treatment, with our coordinate system in the rotating frame. The main thing we need to know is the vector quantity, $\mathbf{A_r} \equiv\ \boldsymbol{\omega} \times \mathbf{r}$ which immediately gives us the scalar potential φ_r as shown above, and from there, the pseudo forces. Furthermore, we saw that the *centrifugal* and the *Euler* force terms are similar to force due to electric field, which does not depend on the velocity of the body in rotating frame. Whereas, the *Coriolis* force term is similar to the force due to magnetic field, which depends on the velocity of the body in rotating frame.

Thus, a positive *charge q* in an electromagnetic field, viewed from an *inertial* (or Newtonian) frame behaves similarly as a particle of *mass m*, viewed from a *rotating* frame.

7. Work & Energy in Rotating Frame

<u>The Work-Energy Theorem:</u> In an inertial frame, we are quite familiar with the *work-energy theorem*[2], which states that the work done by an external force is equal to the change in kinetic energy of the particle considered, i.e.:

$$\int \mathbf{F}_{\text{ext}}.(d\mathbf{r})_{\text{i}} = (\Delta T)_{\text{i}}$$

(where the integral is over the path traversed by the particle)

Here, $(d\mathbf{r})_{\text{i}}$ and $(\Delta T)_{\text{i}}$ are infinitesimal displacement and kinetic energy as seen by the inertial observer.

We can state a similar theorem in a non-inertial frame also, provided we take into account all the pseudo forces as well. We can write,

$$m\mathbf{a}_{\text{r}} = \mathbf{F} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{pseudo}}$$

$$\Rightarrow \quad m\frac{d\mathbf{v}_{\text{r}}}{dt}.(d\mathbf{r})_{\text{r}} = \mathbf{F}.(d\mathbf{r})_{\text{r}}$$

$$\Rightarrow \quad m\mathbf{v}_{\text{r}}.d\mathbf{v}_{\text{r}} = (\mathbf{F}.d\mathbf{r})_{\text{r}}$$

$$\Rightarrow \quad \int (\mathbf{F}.d\mathbf{r})_{\text{r}} = \int_i^f m\mathbf{v}_{\text{r}}.d\mathbf{v}_{\text{r}} = \Delta\left(\frac{1}{2}m{v_{\text{r}}}^2\right) = (\Delta T)_{\text{r}} \qquad \ldots\ldots\ldots(14)$$

where, the left hand side integral is taken over path of motion, and $(\Delta T)_{\text{r}}$ reresents the change in kinetic energy in going from point-i (initial) to point-f (final), as found by the rotating observer [$(d\mathbf{r})_{\text{r}}$ is infinitesimal displacement observed by the rotating observer].

<u>The Relation between Work Done observed from the Two Frames:</u>

Work done as observed by the rotating observer:

$$(dW)_{\text{r}} = (\mathbf{F}.d\mathbf{r})_{\text{r}}$$

$$= [\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{pseudo}}].[(d\mathbf{r})_{\text{i}} - \boldsymbol{\omega} \times \mathbf{r}dt]$$

$$= \mathbf{F}_{\text{ext}}.(d\mathbf{r})_{\text{i}} - \mathbf{F}_{\text{ext}}.\boldsymbol{\omega} \times \mathbf{r}dt + \mathbf{F}_{\text{pseudo}}.(d\mathbf{r})_{\text{r}}$$

$$= (dW)_{\text{i}} - \mathbf{F}_{\text{ext}}.\boldsymbol{\omega} \times \mathbf{r}dt + \mathbf{F}_{\text{pseudo}}.(d\mathbf{r})_{\text{r}}$$

This is the relationship between the work done observed by the rotating observer, and by the inertial observer. Note the significance of the two extra terms on the right hand side:

(i) Note that the term $-\mathbf{F}_{\text{ext}}.\boldsymbol{\omega} \times \mathbf{r}dt$ gives a contribution involving the external force, which was not there in the work expression given by the inertial observer. This factor comes because of the fact that the *velocity* observed by the inertial and the rotating observer are different, for which *trajectory* of the concerned body is also different with respect to the two observers. Indeed, this term is responsible for getting some work done (in the rotating frame), even by the forces which are actually perpendicular to the direction of motion of the body (and hence, are *no-work force*) in the inertial frame. This term actually comes due to the motion of the rotating frame itself.

(ii) The term $\mathbf{F}_{\text{pseudo}}.(d\mathbf{r})_{\text{r}}$ on the right hand side is just the work done by the pseudo forces in the rotating frame.

Note that, the Coriolis force $(2m\boldsymbol{\omega} \times \mathbf{v}_{\text{r}})$ is perpendicular to the velocity $\mathbf{v}_{\text{r}}$ and hence to the rotating frame displacement $(d\mathbf{r})_{\text{r}}$.

Hence, work done by the Coriolis force in the rotating frame:

$$\mathbf{F}_{\text{Coriolis}}.(d\mathbf{r})_{\text{r}} = 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{r}}.(d\mathbf{r})_{\text{r}} = 2m\boldsymbol{\omega} \times \mathbf{v}_{\text{r}}.\mathbf{v}_{\text{r}}dt = 0$$

i.e. Coriolis force has no contribution in the total work done.

Summing up the whole thing, as we go from the inertial to the rotating frame, although the external force remains same, additional contributions from the pseudo forces come. Furthermore, velocity observed from the rotating frame is different from that from inertial frame. For this, *work done* as observed from the non-inertial frame gets modified from that observed from the inertial frame. Again, *kinetic energy* calculated by the non-inertial observer is different from that by inertial observer because of inclusion of the velocity of the non-inertial frame itself. But these two expressions, namely that of the work done and that of the change in kinetic energy *changes in such a way*, that always *equation-(14) holds*. Actually, this has to be so, because, the equation-(14) is a form of statement of *conservation of energy* from the non-inertial observer.

8. Work done in a Loop – Similarity between Euler Force and Force Due to Induced EMF

In electromagnetic case, the electric field is given by:

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial \text{t}}$$

where the 1st term on the right hand side is the conservative part, and the 2nd term is the non-conservative part. We know, this non-conservative part actually comes from changing magnetic field (in fact, change of magnetic flux through the loop considered) and this actually gives rise to the induced emf. Work done in going around a loop, by the electromagnetic forces, as found by the inertial observer:

$$W = q\oint\left(-\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial\, \mathrm{t}} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right).\, d\mathbf{r}$$

$$= q\oint\left(-\frac{\partial \mathbf{A}}{\partial\, \mathrm{t}}\right).\, d\mathbf{r}$$

(where the circular integration is over the loop considered)

(Here, $-\boldsymbol{\nabla}\varphi$ is conservative and hence gives 0 on circular integration over any closed loop. Also, $q\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})$ is the magnetic force which is always perpendicular to the velocity of the charged particle, and hence, is a no-work force.)

$$= \; q\int \boldsymbol{\nabla}\times\left(-\frac{\partial \mathbf{A}}{\partial\, \mathrm{t}}\right).\, d\mathbf{S}$$

(by Stokes' theorem, where the integration is over an area resting on the loop)

$$= \; q\int \left(-\frac{\partial \mathbf{B}}{\partial\, \mathrm{t}}\right).\, d\mathbf{S}$$

Now, for simplicity if we consider a case where the loop is on the (x,y)-plane and the magnetic field is in the z direction, and is a function of time t only, then this expression reduces to:

$$W = \; -q\dot{B}S$$ (where, S is the magnitude of the area resting on the loop)

Now, a similar description[5] can be given in case of a particle moving in a loop, from a rotating frame. Then the work done in loop by the pseudo forces, as seen by the rotating observer, is given by:

$$W = \oint \mathbf{F}_{\mathrm{pseudo}}.\,(d\mathbf{r})_{\mathrm{r}}$$

$$= \oint[-2m\boldsymbol{\omega}\times\mathbf{v}_{\mathrm{r}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r}].\,(d\mathbf{r})_{\mathrm{r}}$$

Now, as was previously shown, the electric force analogue in the rotating case is given by:

$$-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r} = m\left[\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\omega}\times\mathbf{r}|^{2}\right) - \frac{\partial}{\partial\, \mathrm{t}}(\boldsymbol{\omega}\times \mathrm{r})\right]$$

where the 2nd term is similar with $\left(-\frac{\partial \mathbf{A}}{\partial t}\right)$ and can be easily shown to be non-conservative. But, the term $m\boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\omega} \times \mathbf{r}|^{2}\right)$ is conservative and the term $(-2m\boldsymbol{\omega} \times \mathbf{v}_{\mathrm{r}})$ is always perpendicular to $\mathbf{v}_{\mathrm{r}}$. Hence, in the work done around a loop, the only contribution comes from the Euler force term, $(-m\dot{\boldsymbol{\omega}} \times \mathbf{r})$. Calculating it explicitly,

$$W = \oint(-m\dot{\boldsymbol{\omega}} \times \mathbf{r}).(d\mathbf{r})_{\mathrm{r}}$$

Now, if we consider a simple case, where, $\boldsymbol{\omega}$ is along z direction and a function of t only, then the above integral reduces to:

$$W = \oint(-m\dot{\boldsymbol{\omega}} \times \mathbf{r}).(d\mathbf{r})_{\mathrm{r}}$$

$$= \oint -m\dot{\boldsymbol{\omega}}.\mathbf{r} \times (d\mathbf{r})_{\mathrm{r}}$$

$$= -m\dot{\omega} \oint \mathbf{r} \times (d\mathbf{r})_{\mathrm{r}}$$

$$= -2\, m\dot{\omega}S$$

where, S is the magnitude of the area resting on the enclosing loop.

Thus we can see that the Euler force acts in a similar fashion to the non-conservative induced electric field in case of an electromagnetic scenario, and also, $B \cong 2\omega$.

9. Equations Similar to Maxwell's Equations

A very natural question comes after this discussion: is there any set of equations in the rotational case similar to Maxwell's equations in electrodynamics? We shall illustrate below that, for certain assumptions about nature of variation of $\boldsymbol{\omega}$, such a set of equations can indeed be formed. First of all, according to our previous consideration (for the rigidity of the disc) $\boldsymbol{\omega}$ is a **function of time only** (and not of space). Then any partial space derivative acting on $\boldsymbol{\omega}$ produces zero. As we saw previously the electric and magnetic field analogues are:

$$\mathbf{B}_{\mathbf{r}} = 2\boldsymbol{\omega} \quad \ldots\ldots\ldots\ldots\ldots (15a)$$

$$\mathbf{E}_{\mathbf{r}} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \dot{\boldsymbol{\omega}} \times \mathbf{r} \quad \ldots\ldots\ldots\ldots\ldots (15b)$$

which comes from simple calculations.

Now, by simple calculation, remembering the fact that the partial space derivatives produce any nonzero result acting on above field expressions only when they act on the position vector $\mathbf{r}$, using chain rule, we get the following four equations:

$$\boldsymbol{\nabla}.\mathbf{B}_{\mathbf{r}} = 0 \quad \ldots\ldots\ldots (16a) \qquad \boldsymbol{\nabla} \times \mathbf{B}_{\mathbf{r}} = \mathbf{0} \quad \ldots\ldots\ldots (16c)$$

$$\nabla.\mathbf{E_r} = 2\omega^2 \quad \dots\dots\dots (16b) \qquad \nabla \times \mathbf{E_r} = -2\dot{\boldsymbol{\omega}} \quad \dots\dots\dots (16d)$$

Here we note that the non-zero contribution to $\nabla \times \mathbf{E_r}$ comes solely from $-\dot{\boldsymbol{\omega}} \times \mathbf{r}$ part, which is the non-conservative part. The implications of the obtained equations are quite intuitive. Note that all points on the disc are moving with angular velocity $\boldsymbol{\omega}$ (except the central point, which is at rest). Hence the rotating observer will see everywhere a 'magnetic' field $2\boldsymbol{\omega}$. The 'magnetic' field has no spatial dependence; it may change with time, but at a given instant it is same everywhere. Hence it must be solenoidal and irrotational, which is implication of equations (16a) and (16c). Moreover, the conservative part of the 'electric' field is equal to the centrifugal acceleration and hence directed radially outwards, for which it must be irrotational, but have a non-zero, positive divergence. However, the non-conservative part has a non-zero curl (if angular velocity changes with time), though the divergence is zero. This facts are contained in equations (16b) and (16d).

We can go even a step further. Note that in equation-(16b), $2\omega^2$ (=ρ_r, say) can be regarded as '*charge density*'. Equation-(16d) can be rewritten as:

$$\nabla \times \mathbf{E_r} + \frac{\partial \mathbf{B_r}}{\partial t} = \mathbf{0}$$

because, from the definition, $\mathbf{B_r} = 2\boldsymbol{\omega}$.

But the equation-(16c) does not look like Maxwell's equation. Here, we make an assumption that, $\dot{\boldsymbol{\omega}}$ is along $\boldsymbol{\omega}$, i.e., $\boldsymbol{\omega}$ changes only by magnitude, but not in direction. Also, $\ddot{\boldsymbol{\omega}} = \mathbf{0}$. Then, we can write:

$$\frac{\partial \mathbf{E_r}}{\partial t} = \frac{\partial}{\partial t}[-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \dot{\boldsymbol{\omega}} \times \mathbf{r}]$$

$$= -[\dot{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \ddot{\boldsymbol{\omega}} \times \mathbf{r}]$$

$$= -2\mathbf{r}(\dot{\boldsymbol{\omega}}.\boldsymbol{\omega})$$

$$= -2\dot{\omega}\omega\mathbf{r}$$

Now if we define '*current density*', $\mathbf{J_r} \equiv \frac{2}{3}.(-2\dot{\omega}\omega\mathbf{r})$, then we can see, equation-(16c) also takes the form of Maxwell's equation:

$$\nabla \times \mathbf{B_r} + \frac{\partial \mathbf{E_r}}{\partial t} = \frac{3}{2}\mathbf{J_r}$$

(in original Maxwell's equation, a constant μ_0 was there, instead of $\frac{3}{2}$)

Also '*equation of continuity*' is valid under this prescription:

$$\nabla.\mathbf{J_r} + \frac{\partial \rho_r}{\partial t} = -4\dot{\omega}\omega + 4\dot{\omega}\omega = 0$$

Actually, the $\frac{2}{3}$ factor was needed to be introduced for the validity of the continuity equation. Furthermore, ρ_r and $\mathbf{J_r}$ contain ω, $\dot{\omega}$ and $\mathbf{r}$, which actually determine the 'fields' and hence the forces (actually the pseudo forces), in a similar fashion to electrodynamics, where the charges and currents determine all the fields and hence forces.

10. Central Force – An Illustration of Effective Potential

In this section we are going to illustrate use of the rotating frame Lagrangian in describing motion of a body moving under some *central conservative force field*. In such a case, angular momentum of the body remains constant. We assume that its angular velocity is also constant. Therefore, it moves in a circle. Then the rotating observer is attached with the rotating man. We choose the generalised coordinates as the three cylindrical coordinates.

Here, $U_r = -\frac{1}{2}m|\boldsymbol{\omega}\times\mathbf{r}|^{\mathbf{2}} - m\boldsymbol{\omega}\times\mathbf{r}.\mathbf{v}_{\mathrm{r}} = -\frac{1}{2}m\omega^2 r^2$

The rotating frame Lagrangian: $L_{\mathrm{r}} = \frac{1}{2}m\dot{r}^2 - \left(-\frac{1}{2}m\omega^2 r^2 + V\right)$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 - V$$

Note that the expression contains no other coordinates than r. Hence it is sufficient to deal with only r-equation here.

The radial equation of motion is:

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 - V\right)\right\} - \frac{\partial}{\partial r}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 - V\right) = 0$$

or, $m\ddot{r} - m\omega^2 r = -\frac{dV}{dr} = F$

Now as the man is at rest in the rotating frame, then the first term on left hand side of this equation is 0. Thus the equation reduces to:

$$-m\omega^2 r = -\frac{dV}{dr} = F$$

Which just equates the mass times centrifugal force with the central fore, thus keeping the man at rest in the rotating frame. This thing is described from the inertial frame as a rotation due to central force causing the centripetal acceleration (as discussed earlier).

11.Conclusion

We illustrated the two different viewpoints of an inertial and a rotating (non-inertial) observer and how motion of a body can be described by each of them. As an application, we discussed Larmor precession classically. Then we described how a velocity dependent potential associated with the pseudo forces in a rotating frame can be designed and from there drew analogy between description of motion of a body (mass) by a rotating observer and description of motion of a charged particle in an electromagnetic field by an inertial observer, with respect to fields, potentials, forces, energy etc. We also designed a set of equations similar to Maxwell's equations, for the rotating frame case, under some simplifying assumptions about nature of variation of the angular velocity of the rotating frame. Lastly, we illustrated the concept of effective potential further by describing motion under a conservative central force.

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References:

1. N. C. Rana and P. S. Joag, "*Classical Mechanics*",Tata McGraw-Hill Publishing Company Limited, New Delhi, Fourteenth reprint (2004)
2. H. Goldstein, *"Classical Mechanics"* (Second Edition), Narosa Publishing House
3. G. A. Moreno and R. O. Barrachina, "Velocity-dependent Potential of a Rigid Body in a Rotating Frame" , *American Journal of Physics*, **76,** 1146 (2008); doi: 10.1119/1.2982632
4. D. A. Manjarr´es, J. H. William and A. Rodolfo, "Work and energy in rotating systems" , arXiv: **1201.2971v1** [physics.class-ph] 13 Jan 2012
5. M. D. Semon and G. M. Schmieg, "Note on the analogy between inertial and electromagnetic forces" *American Journal of Physics* **49**, 689 (1981); doi: 10.1119/1.12428
6. I. E. Tamm, *"Fundamentals of the Theory of Electricity"* , Mir Publishers, Moscow, First Published English Translation (1979)