

# A Double Pendulum: Fringing into Non-Linearity

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## Abstract

We have investigated in experiment, a variation of the simple double pendulum in which the point masses have been replaced by rectangular plates. We have captured in video, this system in its various states of motion- small oscillatory and chaotic. We have then extracted the frames out of the video using MATLAB. We detected the motion of the two pendulums by identifying the relevant colour pixels in each frame. The pixel co-ordinates were then used to calculate velocities and plot various scaled graphs related to the motion. The fact that in the experiment, the pendulum had actually swung in chaotic motion, has also been demonstrated. We have also done a simulation of the simple double pendulum using MATLAB, where the initial conditions and system dimensions are adjustable by the user. This represents an ideal situation, as all the conditions are controlled. Velocity-time plots and phase portraits have been drawn from the simulation data.

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\*This project was done when the author was at St. Xavier's College, Kolkata, under the supervision of Dr. Shibaji Banerjee.

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### 1. Introduction

The double pendulum is a mechanical device that consists of two rigid pendulums linked together. In the present case, the double pendulum is implemented using two transparent bars connected to each other by screws. It has a rich dynamical behaviour that is governed by a set of coupled ordinary differential equations. The double pendulum undergoes *chaotic motion* for certain energies and is very sensitive to initial conditions.

Poincaré was the first person to glimpse the possibility of *chaos*, in which a deterministic system exhibits aperiodic behaviour that depends sensitively on the initial conditions. Small variations in these conditions produce widely different results, in such a way that the long term behaviour of chaotic systems cannot be predicted. Chaotic behaviour has been observed in the laboratory in a variety of systems including electrical circuits, lasers, oscillating chemical reactions, fluid dynamics, and mechanical and magneto-mechanical devices, as well as computer models of chaotic processes.

A commonly used definition says that, for a dynamical system to be classified as chaotic, it must have the following properties:

- It must be sensitive to initial conditions.
- Its periodic orbits must be dense.

*Sensitivity to initial conditions* means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behaviour. Sensitivity to initial conditions is popularly known as the "*butterfly effect*."

*Density of periodic orbits* means that every point in the space is approached arbitrarily closely by periodic orbits.

Chaos theory became formalised when it first became evident for some scientists that linear theory simply could not explain the observed behaviour of certain experiments like that of the *logistic map*. What had been beforehand excluded as measure imprecision and simple "*noise*" was considered by chaos theories as a full component of the studied systems.

The main catalyst for the development of chaos theory was the electronic computer. Much of the mathematics of chaos theory involves the repeated iteration of simple mathematical formulae, which would be impractical to do by hand. Computers made these repeated calculations practical, while figures and images made it possible to visualise these systems.

## 2. Theory

### 2.1 The Equations of Motion

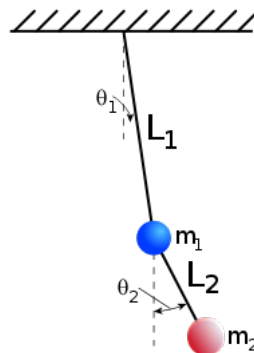


Figure 1: A Double Pendulum

The diagram shows the geometry of the double pendulum.

The equations of motion may be determined from the Lagrangian of the system. The kinetic energy of the device is the sum of the translational kinetic energy of the centre

of mass of each pendulum, and the rotational kinetic energy of each pendulum about its centre of mass. It is given as:

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$\text{i.e. } T = \frac{1}{2}m_1L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2(L_1^2\dot{\theta}_1^2 + L_2^2\dot{\theta}_2^2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)) \quad (1)$$

The potential energy is equivalent to that of two point masses at the locations of the centre of mass of each pendulum. We introduce the following new variables:

$$x_1 = L_1 \sin \theta_1$$

$$y_1 = L_1(1 - \cos \theta_1)$$

$$x_2 = L_1 \sin \theta_1 + L_2 \sin \theta_2$$

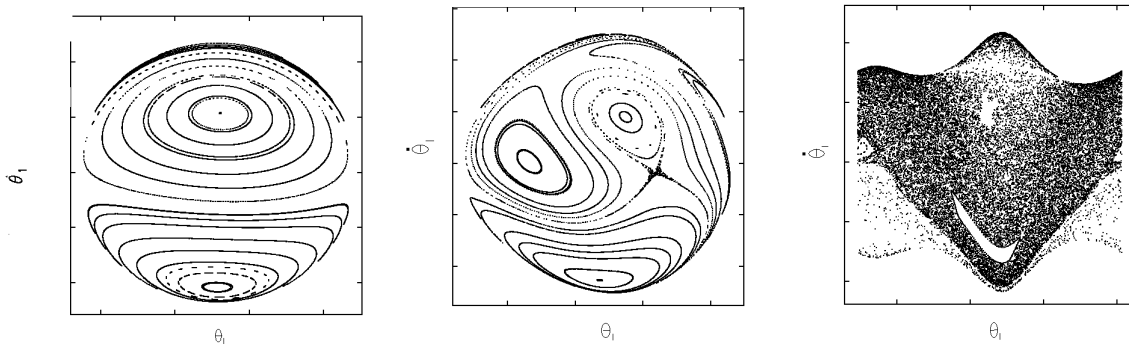
$$y_2 = L_1(1 - \cos \theta_1) + L_2(1 - \cos \theta_2)$$

Thus, we get,

$$V = m_1L_1g(1 - \cos \theta_1) + m_2(L_1(1 - \cos \theta_1) + L_2(1 - \cos \theta_2))g$$

$$\text{i.e. } V = (m_1 + m_2)L_1g(1 - \cos \theta_1) + m_2L_2g(1 - \cos \theta_2) \quad (2)$$

In the above,  $m_1$  and  $m_2$  represent the masses of the two bobs,  $L_1$  and  $L_2$  are the respective effective lengths of the two pendulums,  $\theta_1$  and  $\theta_2$  represent their angular displacements,  $\dot{\theta}_1$  and  $\dot{\theta}_2$  represent the angular velocities and  $g$  is the acceleration due to gravity.



(a) Poincaré Section at Low Energies      (b) Poincaré Section at Medium Energies      (c) Poincaré Section at High Energies

Figure 2: Poincaré Sections for Different Energies

The Lagrangian of the system is given by:

$$\mathcal{L} = T - V$$

From here, we get the equations of motion, or the Euler-Lagrange equations, as follows:

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} \right) - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \quad (3)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} \right) - \frac{\partial \mathcal{L}}{\partial \theta_2} = 0 \quad (4)$$

Written explicitly, they look like this:

$$\dot{\alpha}_1 + \mu_2 \frac{L_2}{L_1} \dot{\alpha}_2 + \cos(\theta_1 - \theta_2) + \mu_2 \frac{L_2}{L_1} \alpha_2^2 + \sin(\theta_1 - \theta_2) + \omega_1^2 \sin \theta_1 = 0 \quad (5)$$

$$\frac{L_2}{L_1} \dot{\alpha}_2 + \dot{\alpha}_1 \cos(\theta_1 - \theta_2) - \alpha_1^2 \sin(\theta_1 - \theta_2) + \omega_2^2 \sin \theta_2 = 0 \quad (6)$$

Here, the following new variables have been used:

$$\begin{aligned} \alpha_1 &= \dot{\theta}_1 \\ \alpha_2 &= \dot{\theta}_2 \\ \mu_2 &= \frac{m_2}{m_1 + m_2} \\ \omega_1^2 &= \frac{g}{L_1} \end{aligned}$$

## 2.2 Poincaré Sections

The general dynamics of the pendulum maybe investigated by analysing the phase space for increasing values of total energy. The phase space of the pendulum is three dimensional. There are four co-ordinates, i.e.  $\theta_1$ ,  $\theta_2$ ,  $\dot{\theta}_1$  and  $\dot{\theta}_2$ , but one of these maybe eliminated because energy is conserved.

The phase space maybe examined by considering the two dimensional Poincaré Section, defined by selecting one of the phase elements and plotting the values of others every time the selected element has a certain value. For a given choice of initial conditions, the Poincaré Section shows points representing the intersection of an orbit in phase space with a plane in phase space. Periodic orbits produce a finite set of points in the Poincaré Section, quasi-periodic orbits produce a continuous curve and chaotic orbits result in scattering of points within an energetically accessible range.

We use conservation of energy to eliminate  $\dot{\theta}_2$  and choose the Poincaré plane to be  $\theta_2 = 0$ .

The total energy is given by:

$$E_{tot} = T + V \quad (7)$$

From this, we get a quadratic equation for  $\dot{\theta}_2$ , with the following solutions:

$$\dot{\theta}_2 = -\frac{L_1}{L_2} \dot{\theta}_1 \cos \theta_1 \pm \sqrt{\left(\frac{L_1}{L_2} \dot{\theta}_1 \cos \theta_1\right)^2 - \frac{2}{m_2 L_2^2} ((m_1 + m_2) g L_1) (1 - \cos \theta_1) + \frac{1}{2} (m_1 + m_2) L_1^2 \dot{\theta}_1^2 - E_{tot}}$$

We now plot a point  $(\theta_1, \dot{\theta}_1)$  in the phase space of the inner pendulum, when the two conditions  $\theta_2 = 0$  and  $\dot{\theta}_2 + \frac{L_1}{L_2} \dot{\theta}_1 \cos \theta_1 > 0$  are fulfilled. This leads to a unique definition of a Poincaré phase space section.

For systems that have low total energies [Figure 2(a)], the motions of the pendulums will not have enough energy to reach large angles. Because of this, the diagrams are more contained. There are more periodic orbits available within the system. With an increase in the total energy of the system [Figure 2(b)], comes more kinetic energy available to each pendulum, so their swings can become greater and their positions will vary more with time. The resultant Poincaré Section is seen to be larger and more complex. There are the beginnings of some chaotic behaviour and multiple stable orbits within the different regions. For systems with larger total energies [Figure 2(c)], the system can be observed to become more chaotic. Because the total energy of the system is so great, neither pendulum is confined within the periodic boundaries. As a result, the Poincaré Section becomes increasingly large and unstable.

### 3. Experimenting with the Compound Double Pendulum

#### 3.1 Capturing the Video

In order to carry out this procedure, the following *instruments* have been used:

- ★ Two rectangular plates made of *aspeX*.
- ★ Bearings
- ★ A vice
- ★ Colour
- ★ A camera
- ★ A stand

The end of the *aspeX* bars were coloured differently so as to be able to identify relevant pixels later on. The two pendulums were attached to each other using bearings such that they were free to oscillate. The inner pendulum was attached to another bar with bearings, which was in turn attached to the vice. After setting the system to motion, the movements of the same were captured in video with the help of a camera. The video thus obtained was converted to a format identifiable by MATLAB.

#### 3.2 Getting Separate Frames from the Video

Once the videos were obtained in MATLAB's recognisable version (with a bitrate of  $24\text{kbps}$  and with *MS-MPEG4 V2* coding). The number of frames per second were adjusted at 100 and the total number of frames thus produced were according to the duration of each video.

#### 3.3 Identifying Colour Pixels from Each Frame and Calculating Positions and Velocities

A code was written which read each frame serially and this was designed to identify the position of the outer pendulum by identifying the *red* colour pixels and the position of the inner pendulum by identifying the *blue* colour pixels.

Since the frame rate was fixed at 100 *fps*, the time difference between two frames was 0.01. By dividing the difference in positions of the pendulums in two consecutive frames by this time difference, the velocities of the two pendulums were obtained.

### 3.4 Results

The plots showing variation of velocity with time and phase plots for motion in small oscillatory range and chaotic range were obtained. It is hard to reach definite conclusions from the velocity-time plots. The basic problem is that it is a multi-dimensional data set: we have four different state variables, plus time.

#### 3.4.1 Small Oscillatory Motion

Figures 3(a), 3(b), 3(c) and 3(d) shows the various plots obtained for small oscillatory motion.

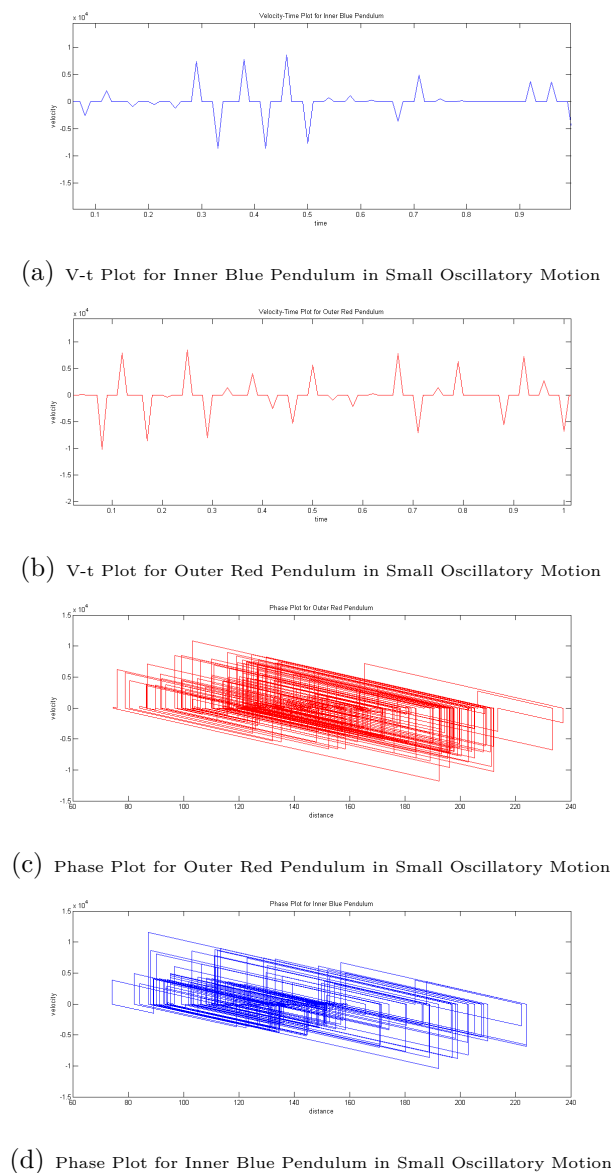
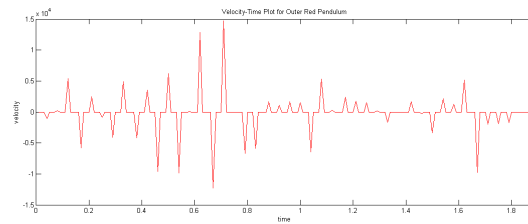


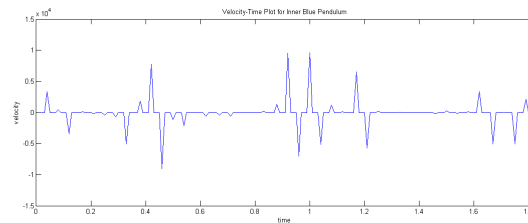
Figure 3: Plots for Small Oscillatory Motion from Experiments

### 3.4.2 Chaotic Motion

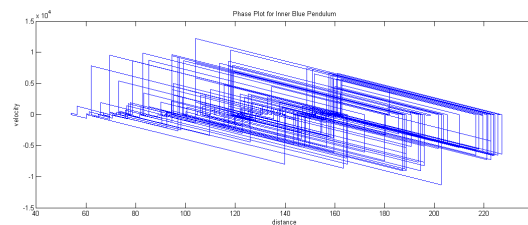
Figures 4(a), 4(b), 4(c) and 4(d) show the various plots obtained for chaotic motion.



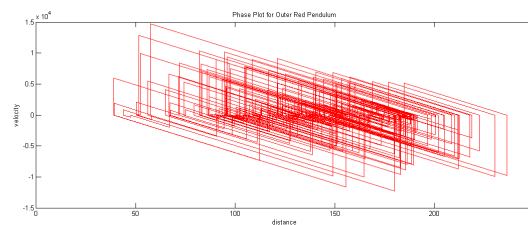
(a) V-t Plot for Outer Red Pendulum in Chaotic Motion



(b) V-t Plot for Inner Blue Pendulum in Chaotic Motion



(c) Phase Plot for Inner Blue Pendulum in Chaotic Motion



(d) Phase Plot for Outer Red Pendulum in Chaotic Motion

Figure 4: Plots for Chaotic Motion from Experiments

When the system oscillates with low total energy, i.e. when it is in small oscillatory motion, the phase space diagrams for both the inner and outer pendulums are observed to be small and compact. When compared to the phase diagrams of the system with larger energy, it can be seen that the phase space is larger and fill up more space. This observation can be attributed to the fact that a system with more energy has more momentum to swing with. The different planes seen in the phase plots correspond to different values of the variables. In the following sub-section, we will show that in these high energies, the pendulum had actually gone on to chaotic range.



### 3.4.3 Histogram of Displacement Data

A histogram was made out of the data for the displacements of the pendulum at different times. The outline of the histogram for the outer red pendulum was seen to be of Gaussian nature. It is shown in Figure 5.

The normal or Gaussian distribution has the following mathematical form:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

where  $\sigma$  is called the standard deviation.

While in chaotic motion, co-relation of the motion of the pendulum with time is apparently destroyed. Hence, its displacements after each time interval represents a *random walk*. Now, the distribution function representing such a case is *Gaussian*, with the peak of the curve having its centre around the most probable density. In Figure 5, it is seen that when the displacements are binned into intervals, they indeed trace the form of a Gaussian curve. Thus, we conclude that the motion of the double pendulum had actually been chaotic.

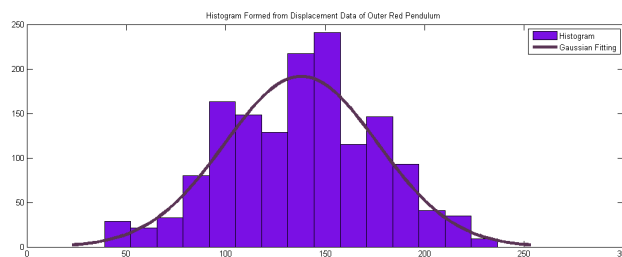


Figure 5: Histogram Formed from Displacement Data of Outer Red Pendulum

## 3.5 Results of Simulation

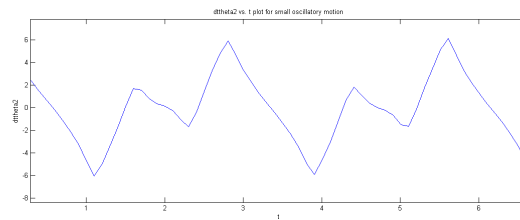
### 3.5.1 Small Oscillatory Motion:

Figures 6(a) and 6(b) show the plots for variation of angular velocity with time for small oscillatory motion.

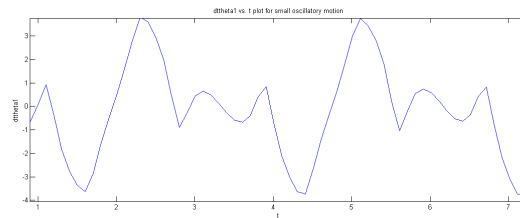
### 3.5.2 Chaotic Motion:

Figures 7(a) and 7(b) show the plots for variation of angular velocity with time for chaotic motion.

We see that in case of the chaotic region, unlike the case of small oscillations, co-relation with time is apparently lost.

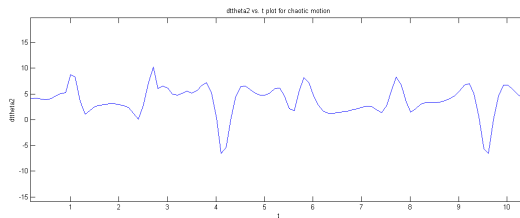


(a) Outer Pendulum, Small Oscillations

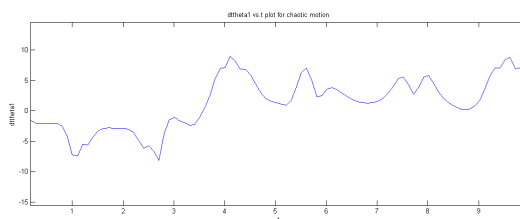


(b) Inner Pendulum, Small Oscillations

Figure 6: Plots for Small Oscillatory Motion under Simulation



(a) Inner Pendulum, Chaotic Motion



(b) Outer Pendulum, Chaotic Motion

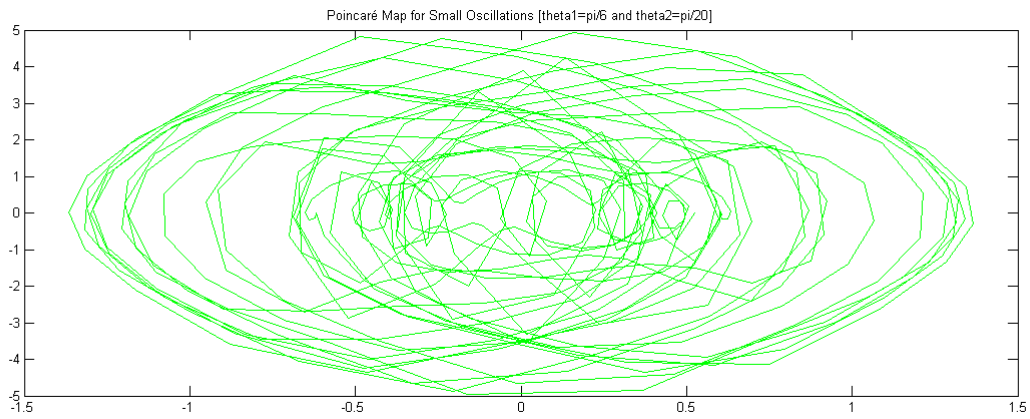
Figure 7: Plots for Chaotic Motion under Simulation

### 3.5.3 Poincaré Sections Obtained from the Simulation:

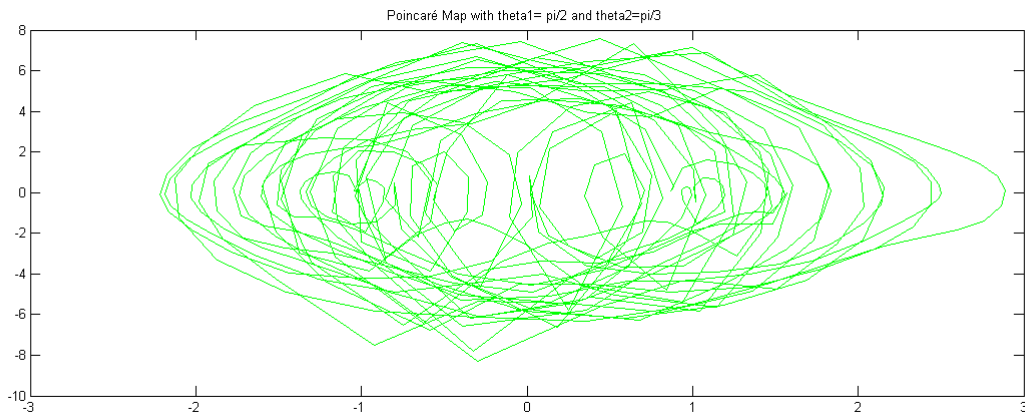
The different Poincaré Sections for different values of  $E_{tot}$  were obtained. With low energy, the obtained Poincaré map is shown in Figure 8(a). Figure 8(a) shows the map where the pendulum begins to move into chaotic range. As clearly seen from the two figures, the phase points spread over the phase plane a lot more in the second case, when compared to the first one.

## 4. Acknowledgements

I would like to sincerely thank the following people for helping me out with several sections of this project, right from the inception of the concept, to arranging for experimental appa-



(a) Poincaré Map for Small Energy



(b) Poincaré Map Showing Beginnings of Chaotic Motion

Figure 8: Poincaré Sections Obtained from Simulation

ratus, to analysis, to techniques, to identification of mistakes along the way, to arranging my thoughts sequentially, to editing and finally, creating something presentable:

- ★ Dr. Shibaji Banerjee
- ★ Dr. Sankha Subhra Nag
- ★ Dr. Suparna Roychowdhury
- ★ Mr. Subrata
- ★ User ImageAnalyst of the forum at [www.mathkb.com](http://www.mathkb.com)
- ★ Mr. Alexander Erlich at *MATLAB Central*.

## Reference Materials

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