Some Initial Concepts of Quantum Information

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1 Qubit

Bit is the unit of information. In case of classical bit (c-bit)only two states 0 and 1 are allowed. But in case of quantum bit (qubit), it is represented by the linear combinaion of the two states as $c_1|0\rangle + c_2|1\rangle$, with the normalisation condition $c_1^2 + c_2^2 = 1$. Therefore infinite number of states are possible. But any pair of orthogonal state will form an orthogonal set. Within an orthogonal set only all the elements can be reliably distinguished from each measurement outcome.

2 Composite System

When a system composed of several qubits interacting (even non-interacting) with each other, is called a composite system. So it can be considered as a closed system composed of n qubits.

2.1 Postulate

When two physical systems are treated as a single combined system, the state space of the combined physical system is the tensor product space $\mathcal{H}1 \otimes \mathcal{H}2$ of the state spaces $\mathcal{H}1, \mathcal{H}2$ of the component subsystems. If the first system is in the state $|\psi 1\rangle$ and the second system is in the state $|\psi 2\rangle$ then the state of the composite system is $|\psi 1\rangle \otimes |\psi 2\rangle$ But not all states of composite may be expressed this way. In general a state of composite system, $|\psi\rangle$ may be expressed as $|\psi\rangle = \sum C_{ij}|\phi_i^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle$

2.2 Entanglement

The state of a two qubit composite system when they are allowed to interact, cannot always be written in the product form. In that case the qubits are said to be entangled. Suppose we have a 2-qubit system and we apply X operation on the first qubit, then this operation maps the system to $X|\psi 1\rangle \otimes I|\psi 2\rangle = (X \otimes I)(|\psi 1\rangle \otimes |\psi 2\rangle$ i.e the linear operator describing this operation on the composite system is $X \otimes I$. If the system is the composition of 'n' qubits, then applying the X gate to the first qubit corresponds to the operation $X \otimes I \otimes I ... \otimes I$ (with n-1 times repeatation of I)to the entire system.

2.3 Pure State and Mixed State

When the state of a system has a definite state vector, the state is called pure state. There are some situations when the state vector of the system cannot be specified with certainity, but the state of a qubit is described by one of a specific set of state vectors with corresponding probabilities. e.g suppose we know that a qubit is in the pure state $|\psi_1\rangle$ with probability 1/3 and is in pure sate $|\psi_2\rangle$ with probability 2/3. The state described by such a probability distribution is called a mixture of states $|\psi_1\rangle$ and $|\psi_2\rangle$. The state of a system in such a situation is called a mixed state. One way of representing a general mixed state on 'n' qubits is as the ensemble

$$\{(|\psi_1\rangle.p_1).(|\psi_2\rangle.p_2)....(|\psi_k\rangle.p_k) \tag{1}$$

which means that the system is in the pure (n-qubit)state $|\psi_i\rangle$ with probability p_i, for i=1,2,...k. So a pure state can be considered as a limiting case where all but one of the p_i's is equal to zero.

2.4 Density Operator

The situation becomes complicated when we use equation (1) in calculation. The alternative representation of mixed states is in terms of operators on the Hilbert space. These are called density operators, and the matrix representation is called a density matrix. The density operator for a pure state state $|\psi\rangle$ is defined as

$$\rho = |\psi\rangle\langle\psi| \tag{2}$$

The probability of getting 0 in terms of density operator is

$$\langle 0|\psi\rangle\langle\psi|0\rangle = \langle 0|\rho|0\rangle \tag{3}$$

The density operator for ensemble of pure states such as eqn (1)is

$$\rho = \Sigma p_i|\psi_i\rangle\langle\psi_i| \tag{4}$$

and contains all the rlevant information about the state of the system. If we apply the unitary operator U to mixed state described in equation (1), we would get the mixed state

$$\{(U|\psi_1\rangle, p_1), (U|\psi_2\rangle, p_2), ...(U|\psi_k\rangle, p_k)\} \tag{5}$$

which has density operator

$$\Sigma p_i U|\psi_i\rangle\langle\psi_i|U^+ = U\rho U^+ \tag{6}$$

2.5 Partial Trace

One important use of density operator formulation is as a tool for describing the state of a subsystem of a composite system. When the composite system is in pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ the system may be or may not be in entangled state. When it is in entangled state, it is not possible to factor out the state vector $|\psi\rangle_A \in \mathcal{H}_A$ for the state of the first qubit. But, from the condition of entanglement it will be shown that the state of the first qubit in general can be described as mixed state. This state can be described by density operator ρ^A on $\mathcal{H}_A$, sometimes it is called reduced density operator as is obtained from the reduction of density operator ρ^{AB} of the composite system. And this is obtained by exploiting the partial trace operation. For a 2-qubit composite system ρ^A is defined in terms of the density operator ρ^{AB} as

$$\rho^A \equiv Tr_B(\rho^{AB}) \tag{7}$$

where Tr_B is the partial trace over system B. Let $|a_1\rangle$ and $|a_2\rangle$ are the bases for the system A and $|b_1\rangle$ and $|b_2\rangle$ are the bases for the system B. Then it can be shown that the bases for the composite AB will be $|a_1b_1\rangle, |a_1b_2\rangle, |a_2b_1\rangle and |a_2b_2\rangle$. Now,

$$Tr_B(|a_1b_1\rangle\langle a_2b_2|) = Tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2| \equiv |a_1\rangle\langle a_2|Tr(|b_1\rangle\langle b_2|) \tag{8}$$

$$= |a_1\rangle\langle a_2|\langle b_2|b_1\rangle \tag{9}$$

The operation of computing Tr_B sometimes is called tracing-out system B. When the composite contains more than two qubits, reduced density operator can be calculated in analogous way. When a bipartite system is expressed in terms of Schmidt basis, e.g let the pure state of the composite AB is $|\psi\rangle$,

$$|\psi\rangle = \sum_i \sqrt{p_i}|\phi_i^A\rangle|\phi_i^B\rangle \tag{10}$$

where $\{\phi_i^A\}$ is a basis for $\mathcal{H}_A$ and $\{\phi_i^B\}$ is a basis for $\mathcal{H}_B$. Then it can be shown that

$$Tr_B|\psi\rangle\langle\psi| = \sum_i p_i|\phi_i^A\rangle\langle\phi_i^A| \tag{11}$$

Similarly,

$$Tr_A|\psi\rangle\langle\psi| = \sum_i p_i|\phi_i^B\rangle\langle\phi_i^B| \tag{12}$$

3 Bell States

In case of two-qubit composite, the states which are maximally entangled are the Bell states. These states are $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle|$; $\beta_{01}\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle|$; $\beta_{10}\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle|$; $\beta_{11}\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$. They are maximally entangled because, if we make any measurement on first qubit (or second qubit), we get entire information about the second (or first) measurement, which is not possible for any other possible states apart from these four states.