Non-linear Dynamical Systems:

An Overview

Sankhasubhra Nag

sankhasubhra_nag@yahoo.co.in

Department of Physics

St. Xavier's College (Autonomous)

Kolkata

Nonlinear Dynamical Systems

In broader sense any time varying system may be considered as Dynamical Systems.

If the time variation is governed by the nonlinear differential equations, the system is called as non-linear dynamical systems. But in physics a host of pivotal differential equations,

Nonlinear Dynamical Systems

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But in physics a host of pivotal differential equations, such as the Maxwell's Equations

$$\nabla.\vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla.\vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \vec{\jmath} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Nonlinear Dynamical Systems

In broader sense any time varying system may be considered as Dynamical Systems.

. . . and the Schrödinger Equation,

$$i\hbar\frac{\partial\psi}{\partial t} = H\left(x, -i\hbar\frac{\partial}{\partial x}\right)\psi$$

are linear in nature

Nonlinear Dynamical Systems

In broader sense any time varying system may be considered as Dynamical Systems.

But our old friend Newton's Law,

$$m\ddot{x} = -F(x, \dot{x}),$$

is not linear in general as all systems are not harmonic oscillators i.e. not always $F(x) = -kx$.

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But our old friend Newton's Law,

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is not linear in general as all systems are not harmonic oscillators i.e. not always $F(x) = -kx$. But any mechanical system near stable equilibrium configuration may be approximated by a harmonic oscillator, i.e. if $x = x_0$ is the stable equilibrium point then for $x = x_0 + \epsilon$,

$$m\ddot{\epsilon} = F(x_0 + \epsilon) \approx F'(x_0)\epsilon,$$

for small ϵ

Pendulum which is not simple

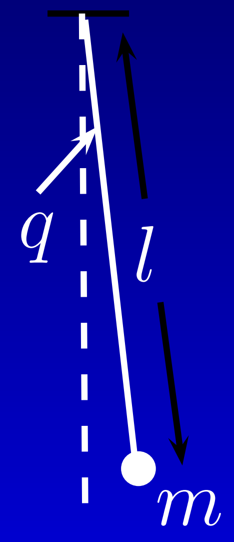
But if ϵ is not so small the terms with higher power will appear in Taylor series and the equation will become non-linear.

Pendulum which is not simple

Let us consider a pendulum, for which the equation becomes, after making q and t dimensionless,

$\ddot{q} = -\sin(q)$

for small amplitude motion about its point of equilibrium.



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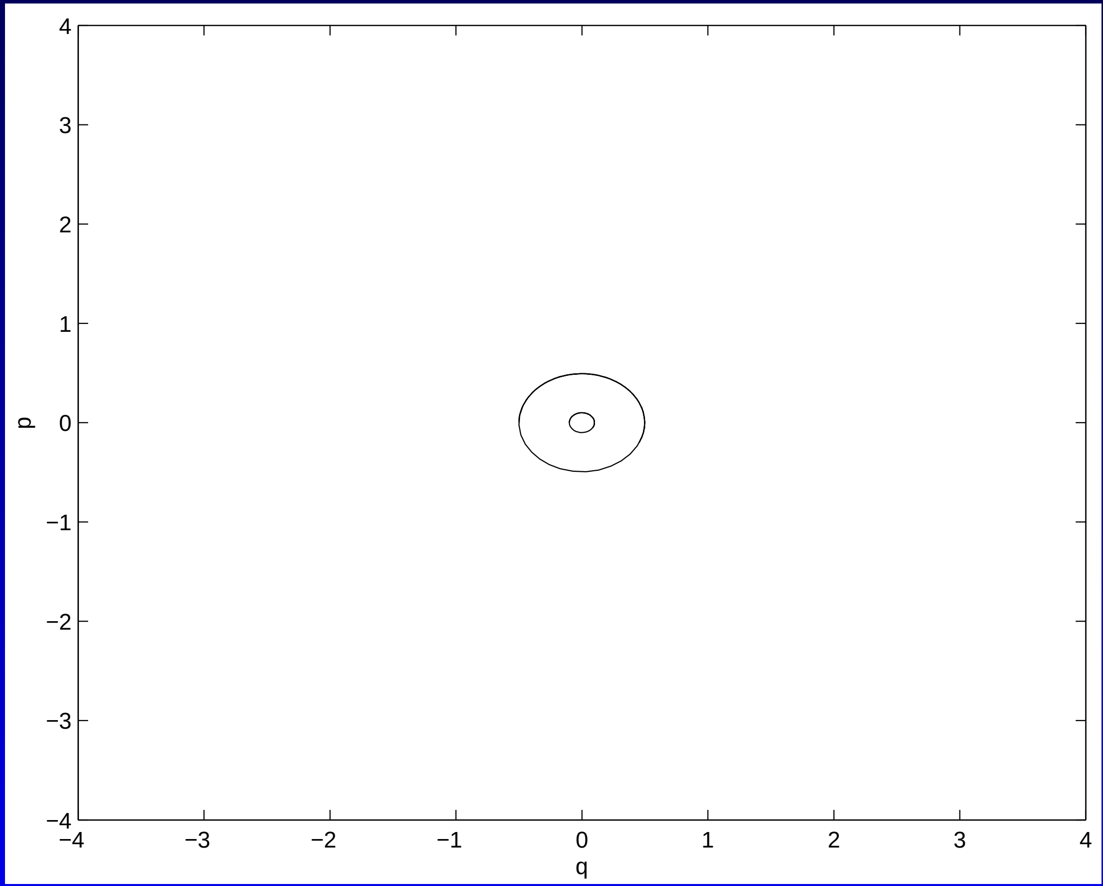

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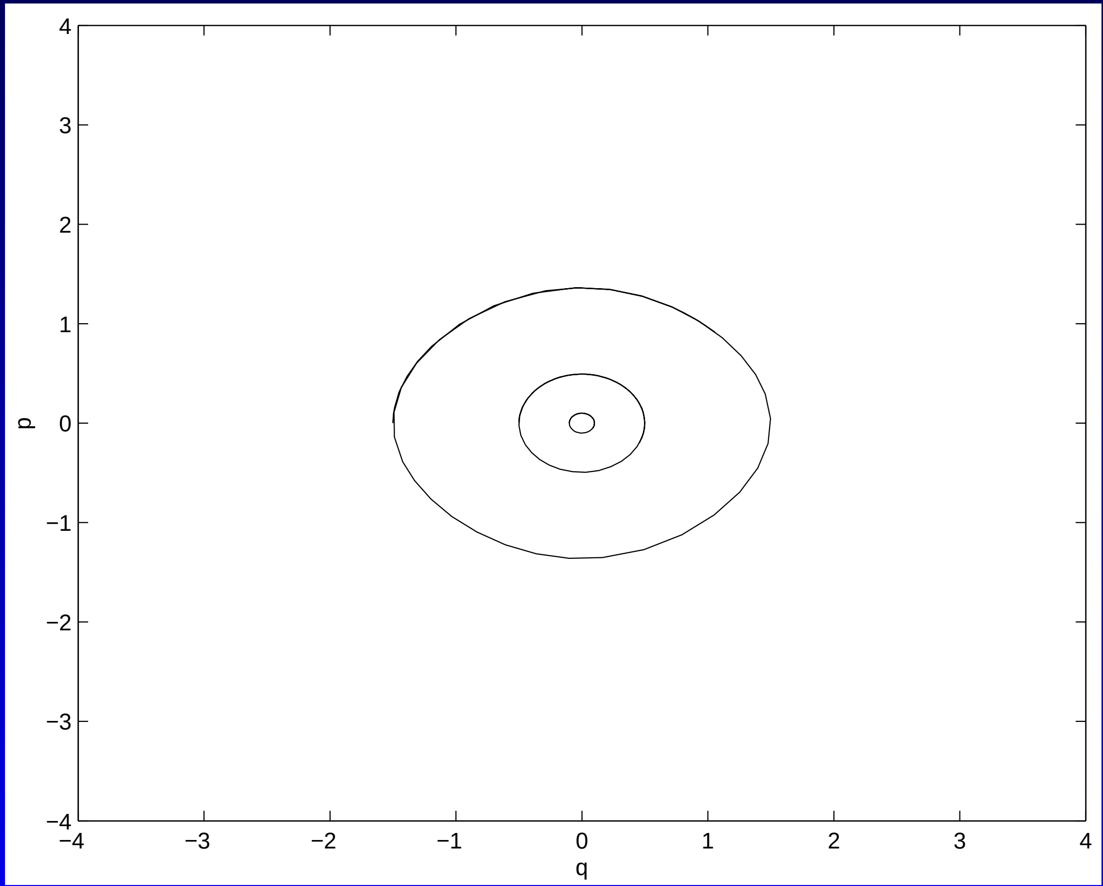

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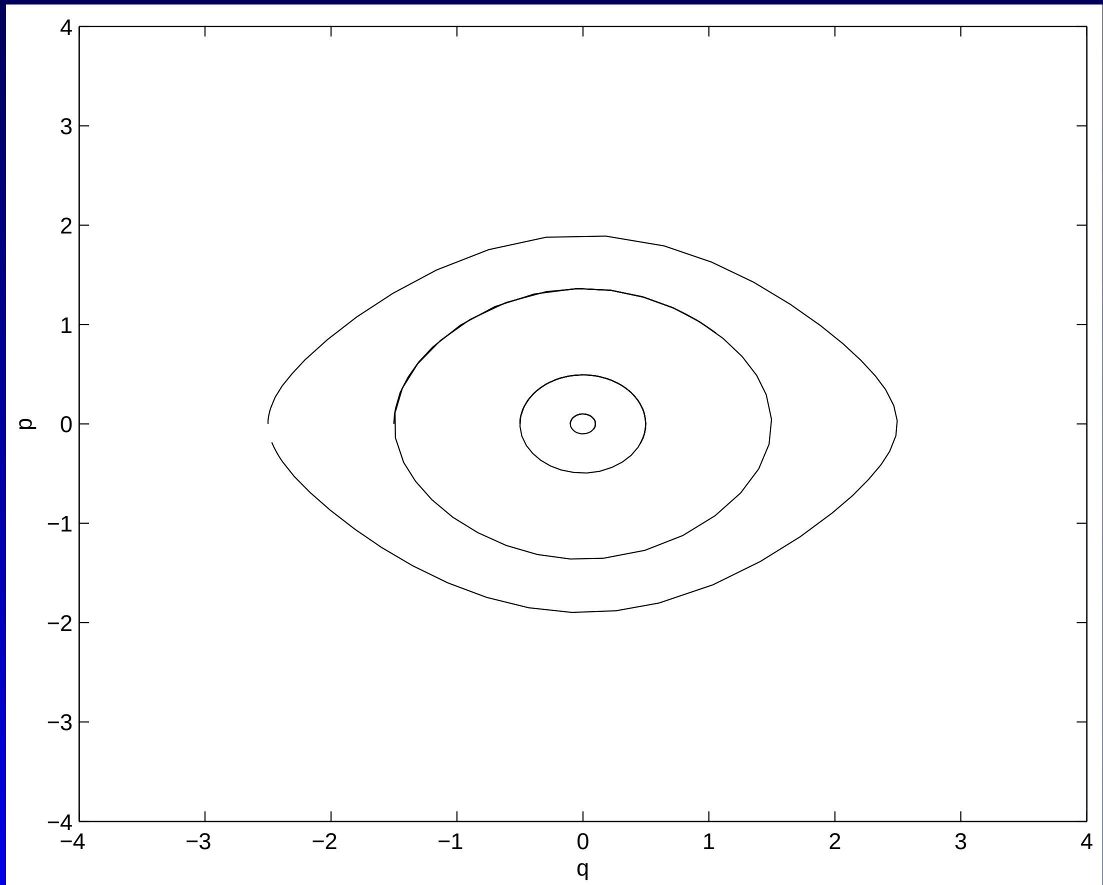

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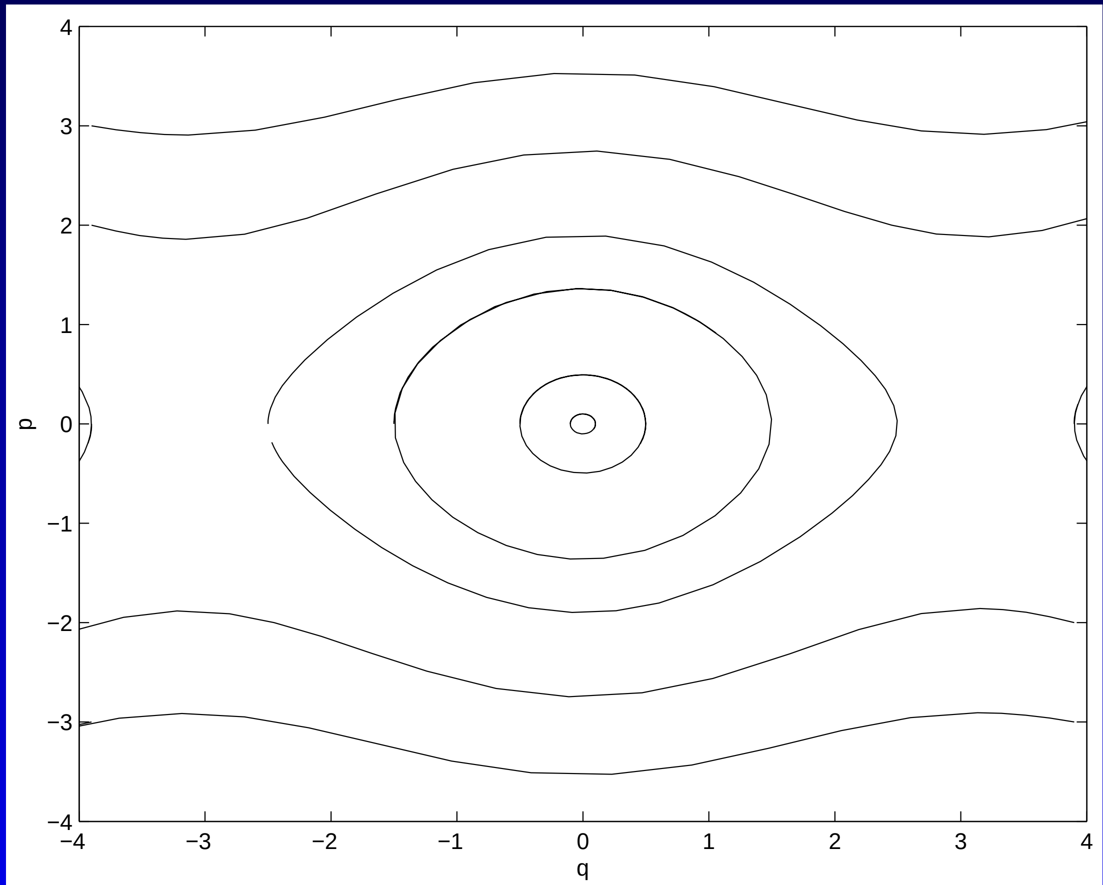

General Method: Fixed Point Analysis

If there is a set of differential equations

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

There are certain points on phase plot which does not move in time and they are called fixed points.

They are the real roots of simultaneous equations $f_1(x_1, x_2) = f_2(x_1, x_2) = 0$

they are denoted by (x_1^*, x_2^*)

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The nearby trajectories of fixed points can be traced by following way 1. First of all, linearize the equations at neighbourhood of fixed points i.e. at $(x_1^* + \epsilon_1, x_2^* + \epsilon_2)$.

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2. The linearized equations are

$$\dot{\epsilon}_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_* \epsilon_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_* \epsilon_2$$
$$\dot{\epsilon}_2 = \left.\frac{\partial f_2}{\partial x_1}\right|_* \epsilon_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_* \epsilon_2$$

General Method: Fixed Point Analysis

If there is a set of differential equations

$$\dot{x}_1 = f_1(x_1, x_2)$$
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The eigenvalues of the stability matrix

$$\underline{M} = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_* & \left.\frac{\partial f_1}{\partial x_2}\right|_* \\ \left.\frac{\partial f_2}{\partial x_1}\right|_* & \left.\frac{\partial f_2}{\partial x_2}\right|_* \end{bmatrix}$$

λ_1 and λ_2 determine the nearby trajectories.

General Method: Fixed Point Analysis

If there is a set of differential equations

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

1. For λ_1, λ_2 real and positive, all nearby trajectories diverge from fixed point.
2. For λ_1, λ_2 real and negative, all nearby trajectories converge to fixed point.
3. For one is positive and other is negative, all nearby trajectories look like hyperbola (saddle).

General Method: Fixed Point Analysis

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$$\dot{x}_1 = f_1(x_1, x_2)$$
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1. For λ_1, λ_2 complex conjugates with positive real part, nearby trajectories look like outward spirals.
2. For λ_1, λ_2 complex conjugates with negative real part, nearby trajectories look like inward spirals.
3. For λ_1, λ_2 purely imaginary conjugates, nearby trajectories look like ellipse. (centre)

Back to pendulum

The equations are $\dot{q} = p, \ \dot{p} = -\sin q$

Fixed points are $(n\pi, 0)$

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$$\underline{M} = \begin{bmatrix} 0 & 1 \\ (\text{-}1)^{n+1} & 0 \end{bmatrix}$$

Back to pendulum

The equations are $\dot{q} = p, \ \dot{p} = -\sin q$

Fixed points are $(n\pi, 0)$

For 0 or even n, $\lambda_{1,2} = \pm i$ i.e. they are centre

For odd n, $\lambda_{1,2} = \pm 1$ i.e. they are saddle

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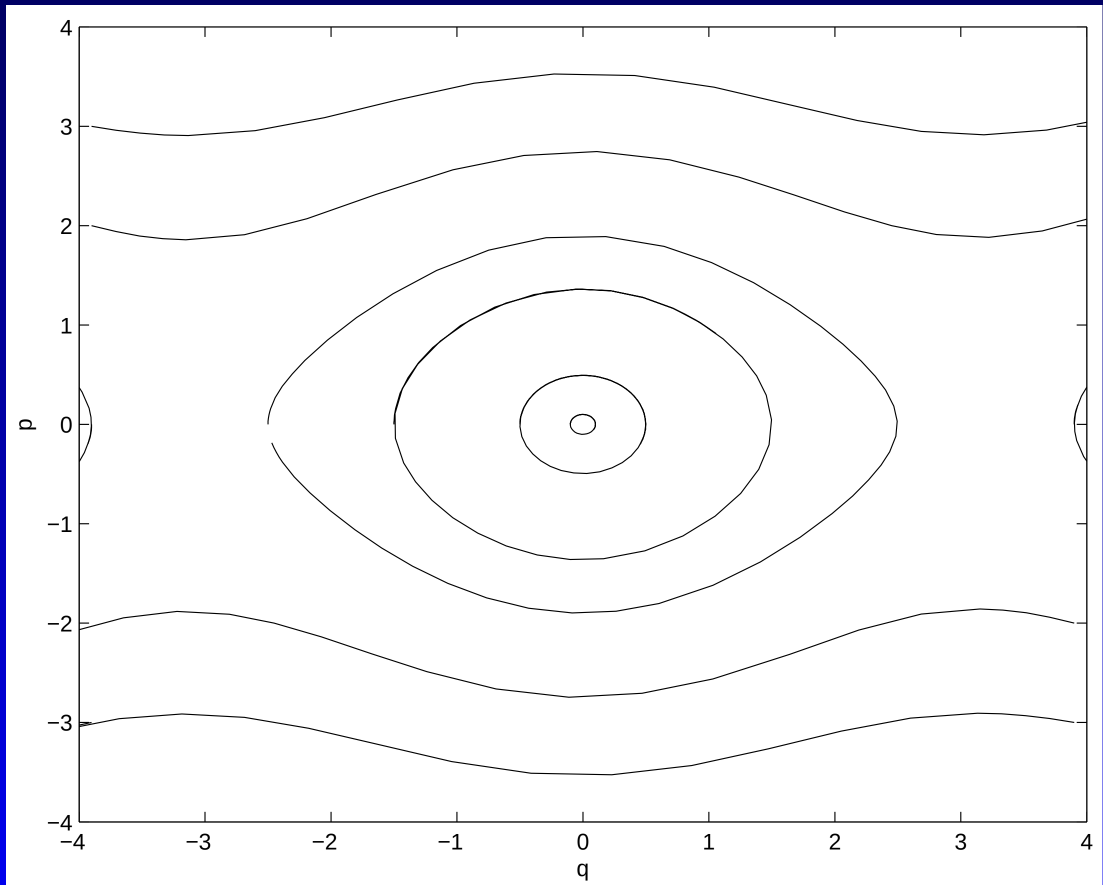

Other exciting non-linear phenomena

1. Bifurcation: With the variation of control parameters qualitative change of phase plot across a certain critical value.

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Other exciting non-linear phe-nomena

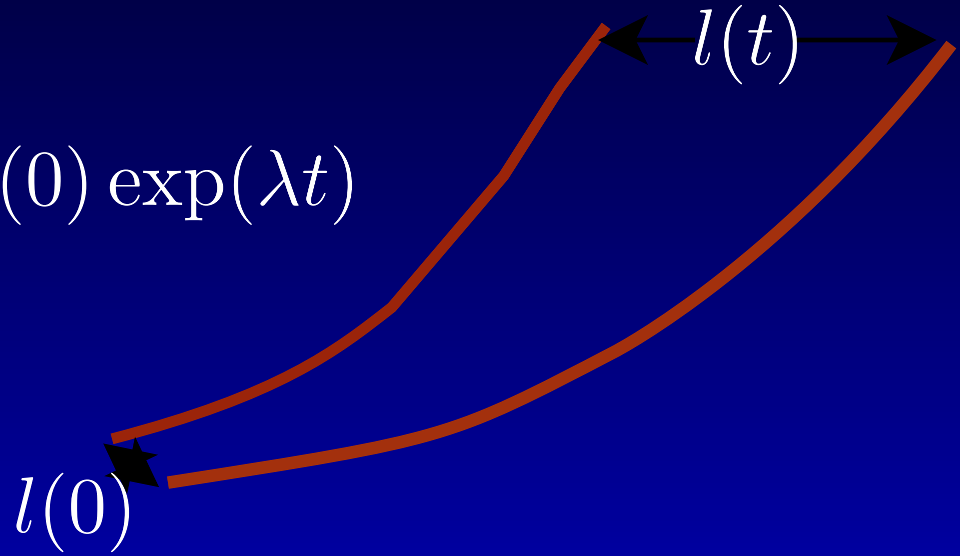
1. Bifurcation: With the variation of control parameters qualitative change of phase plot across a certain critical value.
2. Limit Cycle: Existence of an isolated periodic orbit which attract all nearby trajectories resulting oscillatory behaviour.
3. Solitary Waves: The existence of non-linear term in wave equation may resist the spreading of wave packet in a dispersive medium
4. Chaos: Instability of motion may lead to loss of long time predictability.

Chaos

Some bound motion extremely sensitive to initial conditions is called chaotic.

i.e.

$$l(t) = l(0)\exp(\lambda t)$$



λ is called the Lyapunov exponent, which is a measure of chaos

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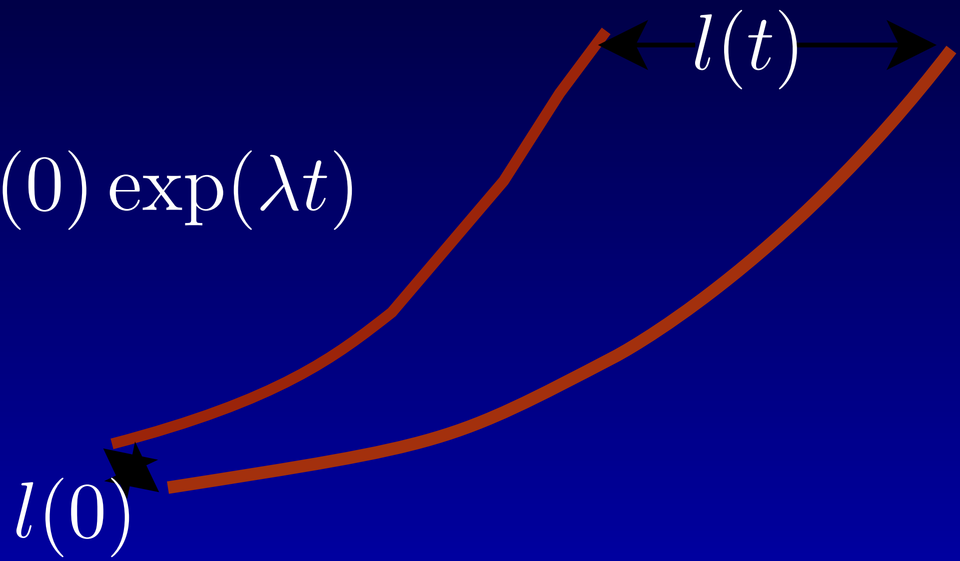
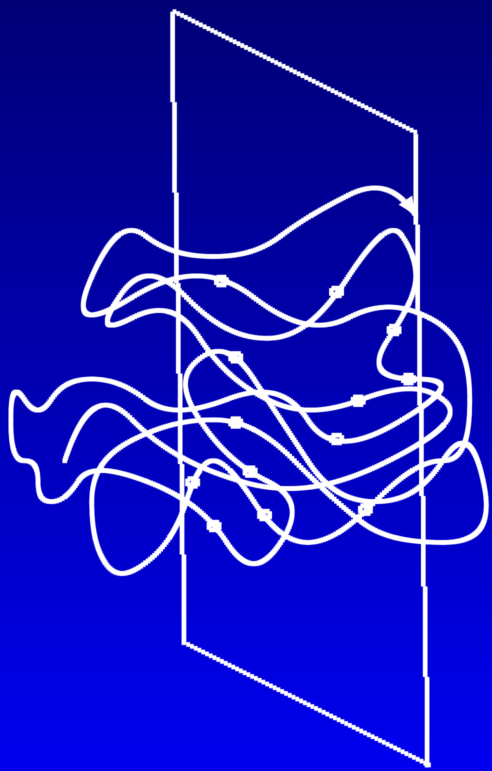


Though chaos is a general concept valid for any type of dynamical system, for the present context we restrict ourselves within the motion of particles under a force field i.e. within the **Hamiltonian systems**.

Chaotic trajectories

For time independent Hamiltonian the trajectories are restricted within the constant energy subspace but –

The trajectories wander around the accessible phase space with various twists and turns to *diverge* away from the vicinity of the others



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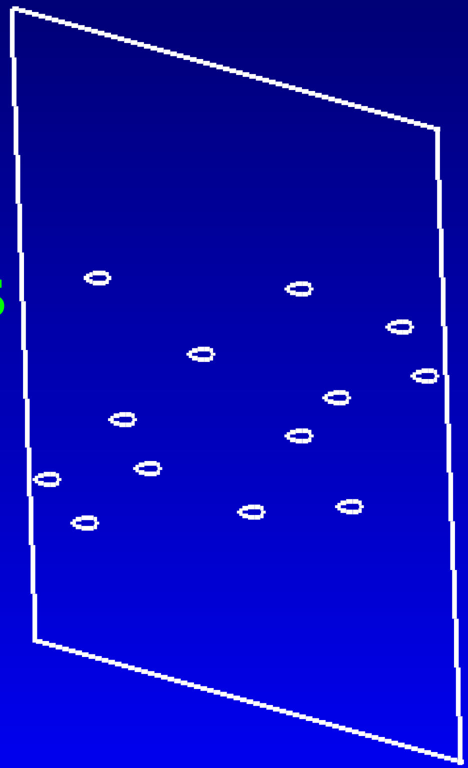
For time independent Hamiltonian the trajectories are restricted within the constant energy subspace but –

the crosssection of the trajectories in lower dimensional plane thus generates scattered points without any pattern characteristic to chaos



Integrable Hamiltonian Systems

Hamiltonian systems are called integrable only when there exist N independent constants of motion

$$f_m(q,p) = c_m \quad m = 1, ..., N$$

and also

$$[f_m, f_n] = \sum_i^N \frac{\partial f_m}{\partial p_i}\frac{\partial f_n}{\partial q_i} - \sum_i^N \frac{\partial f_m}{\partial q_i}\frac{\partial f_n}{\partial p_i} = 0$$

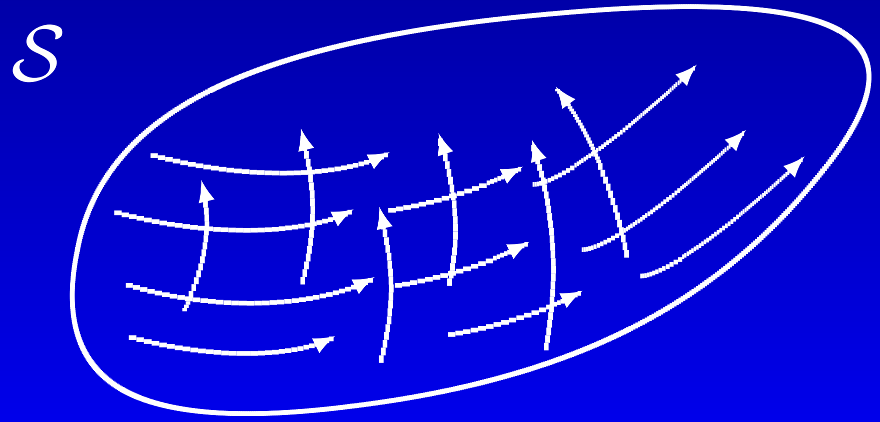
for all $m = 1, ..., N$ & $n = 1, ..., N$

Integrable Hamiltonian Systems

The first condition ensures that for bound motion the trajectories must be restricted within a N dim. closed surface ($\mathcal{S}$); *but not only that,* the second condition ensures that there exist N independent vector fields

$$\underline{V_m} = \left(\frac{\partial f_m}{\partial p_i}, -\frac{\partial f_m}{\partial q_i} \right),$$

each parallel to $\mathcal{S}$ at each point.



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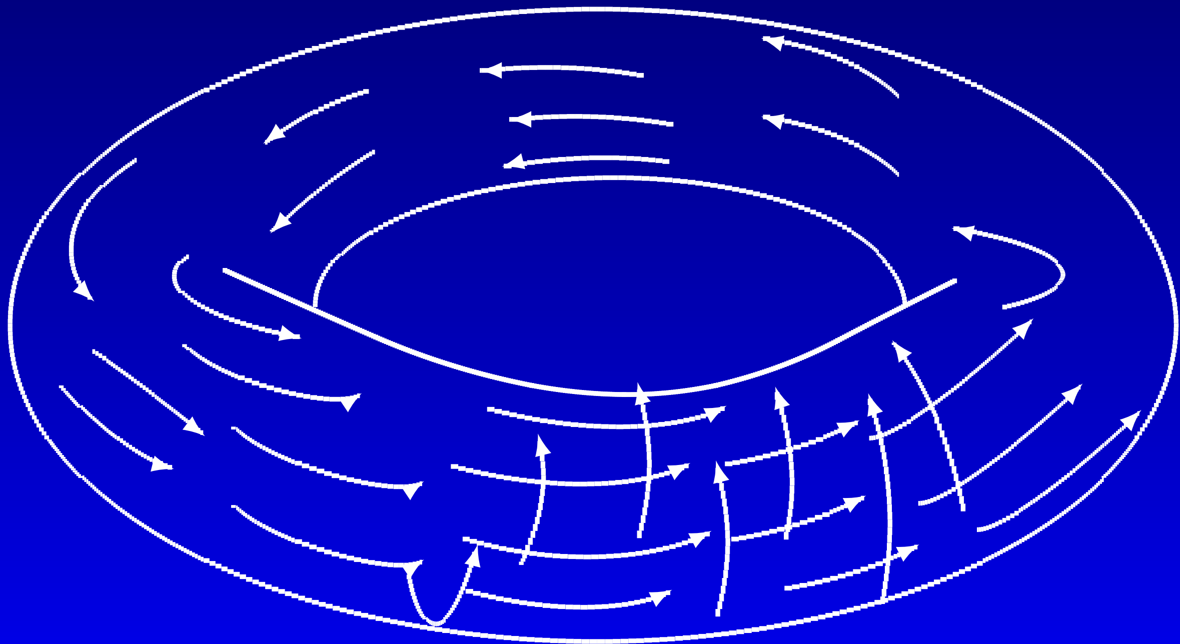


which leaves no option but being a N dim. torus for $\mathcal{S}$.

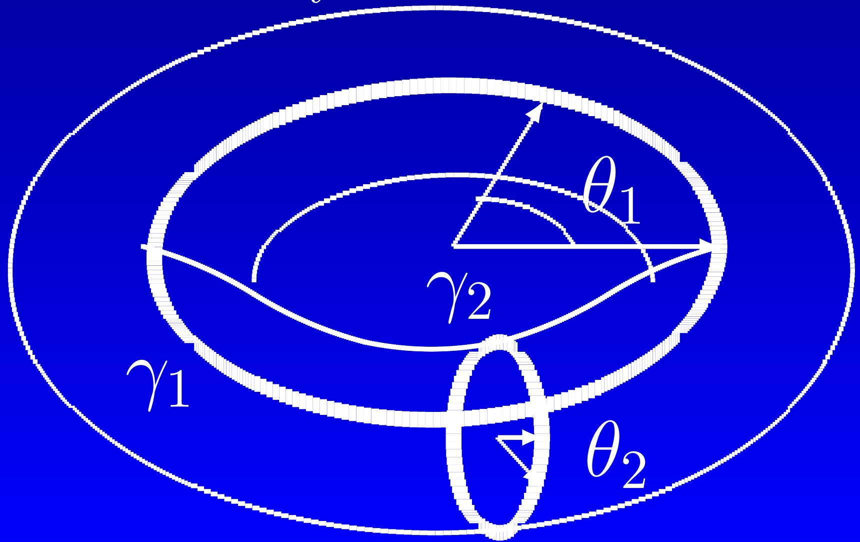
Motion in Phase Space

For the canonical transformation,

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} \theta \\ J \end{pmatrix}$$

where

$$2\pi J_k = \sum_i \oint_{\gamma_k} p_i dq_i = J_k(f = c)$$



Motion in Phase Space

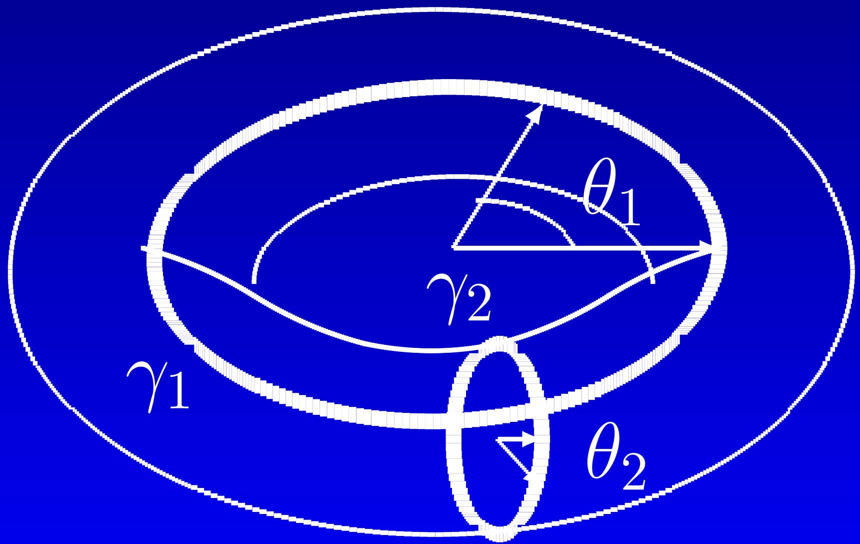
Such that

$$H \longrightarrow H(J)$$

and

$$\dot{J}_k = 0,$$

$$\dot{\theta}_k = \frac{\partial H}{\partial J_k} = \omega_k, const.$$



Motion in Phase Space

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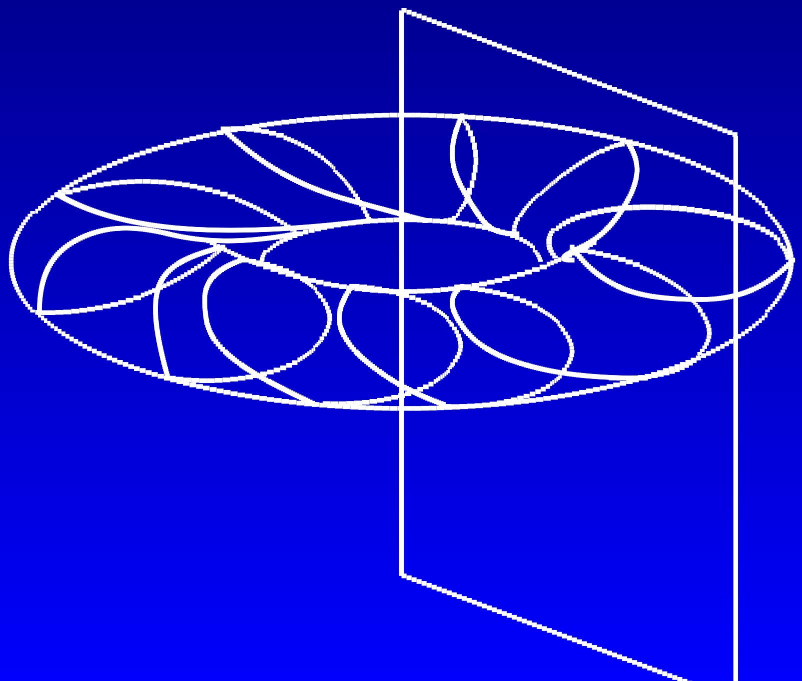
$$H \longrightarrow H(J)$$

and

$$\dot{J}_k = 0,$$

$$\dot{\theta}_k = \frac{\partial H}{\partial J_k} = \omega_k \Rightarrow \theta_k = \omega_k t + \delta_k$$



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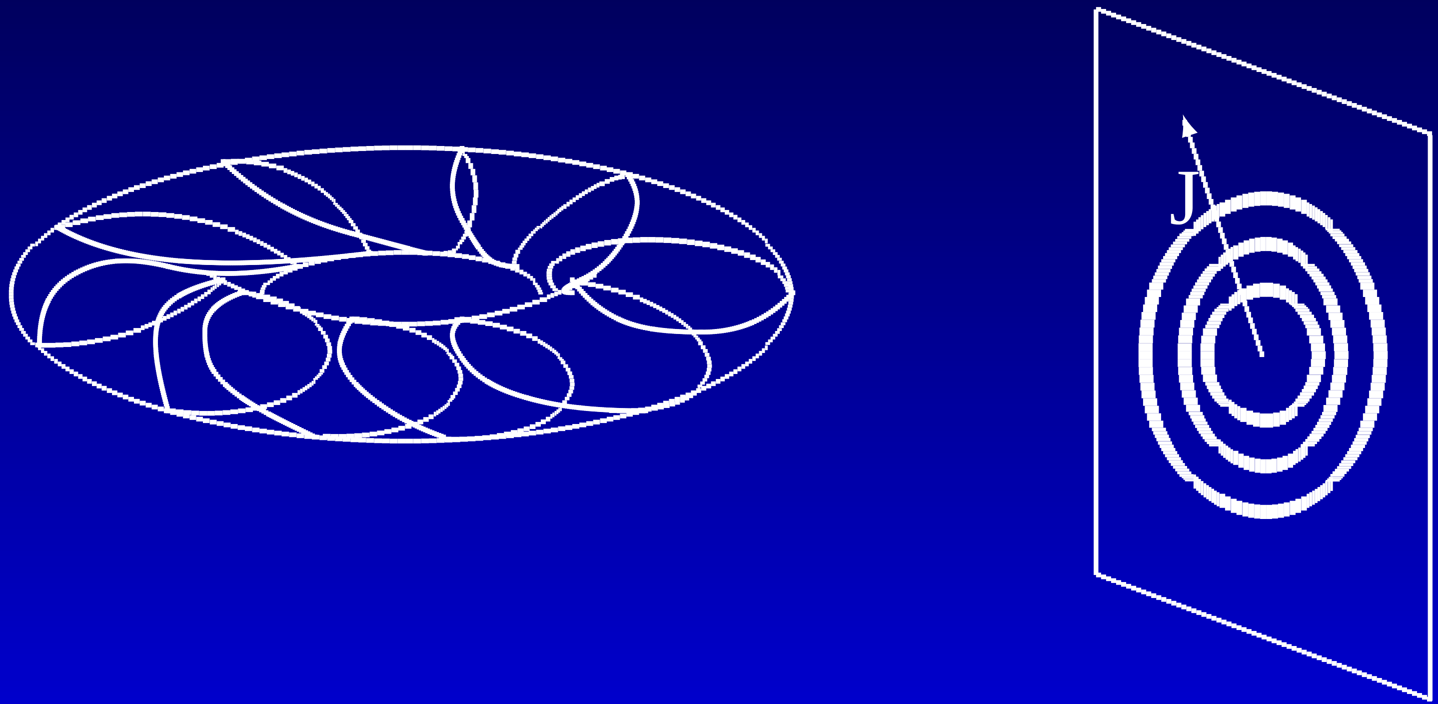

Chaos is generic

As the integrability demands a host of conditions to be maintained simultaneously the regular motion becomes rarer as the degrees of freedom increases.

Even the three body central interaction leads to chaos!

Double pendulum system is chaotic!

and etc. etc. ....

The End

Thank You!