Green’s Functions in Quantum Chaos

Semiclassical Approach

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Introduction

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But such methods collapse for chaotic systems where the motion is exponentially sensitive to initial conditions and thus lose long time correlation.

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Thus new approach is required to tackle this problem.

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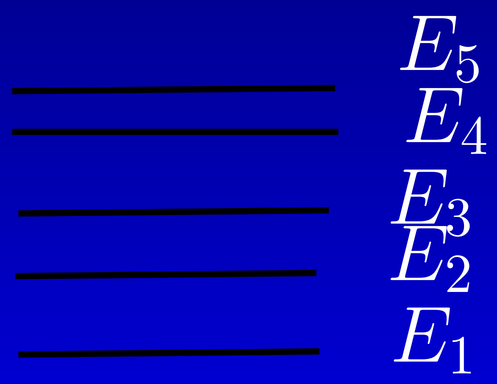

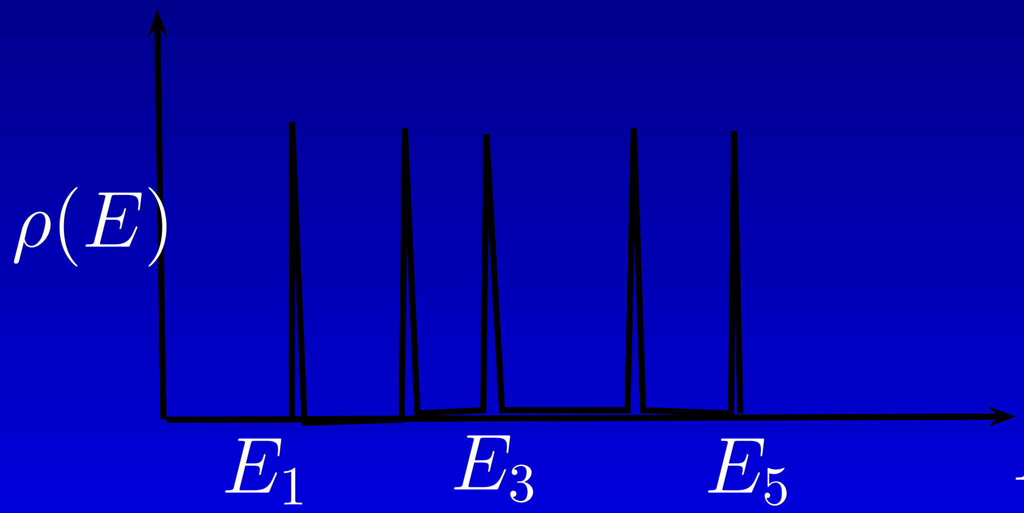

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$$= \lim_{\epsilon \to 0} \Im \left(\sum_n \frac{1/\pi}{(E - E_n) + i\epsilon} \right) \approx \Im \left[\frac{1}{\pi} \mathrm{Tr} \left(\frac{1}{E - \hat{H}} \right) \right]$$

Green's Function

The quantum mechanical Green's function $G(q, q', E)$ is given by :

$$G(q, q', E) = \langle q | \hat{G}(E) | q' \rangle ,$$

where,

$$(E - \hat{H})\hat{G} = \hat{\mathcal{I}}$$

i.e.

$$[E - H(q, \partial/\partial q)]\, G(q, q', E) = \delta(q - q')$$

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where, $|\phi_n\rangle$ represents n'th stationary state.

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Thus,

$$\rho(E) = \Im\left[\frac{1}{\pi}\int G(q, q, E) dq\right]$$

Feynman Propagator

Feynman propagator for a quantum system may defined by the equation :

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right) K(q, q', t) = -i\hbar\delta(t)\delta(q - q')$$

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Multiplying both sides by $\frac{i}{\hbar}\exp(iEt/\hbar)$ and then integrating with respect to t, we get,

$$[E - H(q, \partial/\partial q)]\, g(q, q', E) = \delta(q - q')$$

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is nothing but $G(q,q',E)$ already discussed

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$$g(q, q', E) = \frac{i}{\hbar}\int K(q, q', t)\exp(iEt/\hbar)dt,$$

$$K(q, q', t) = \langle q|\,\Theta(t)\exp\left(-\frac{i\hat{H}t}{\hbar}\right)|q'\rangle$$

where $\Theta(t)$ is Heaviside step function.

Integral Formulation

For very small time interval t,

$$K(q_B, q_A, t) = \left(\frac{1}{2\pi i\hbar}\right)^{d/2} \left|-\frac{\partial^2 W_{BA}}{\partial q_A \partial q_B}\right|^{1/2} \exp\left(\frac{i}{\hbar} W_{BA}\right)$$

where

$$W_{BA} = \int_0^t L dt$$

Integral Formulation

For large time interval, it is divided into an infinite number of subintervals and then integrated over all possible paths between q_A and q_B to get

$$K(q_B, q_A, t) = \lim_{N\to\infty} \left(\frac{1}{2\pi i\hbar}\right)^{Nd/2} \int dq_1 \ldots dq_{N-1}$$

$$\times \left|\prod_i D_{i,i+1}\right|^{1/2} \exp\left(\frac{i}{\hbar}\sum_i W_{i,i+1}\right)$$

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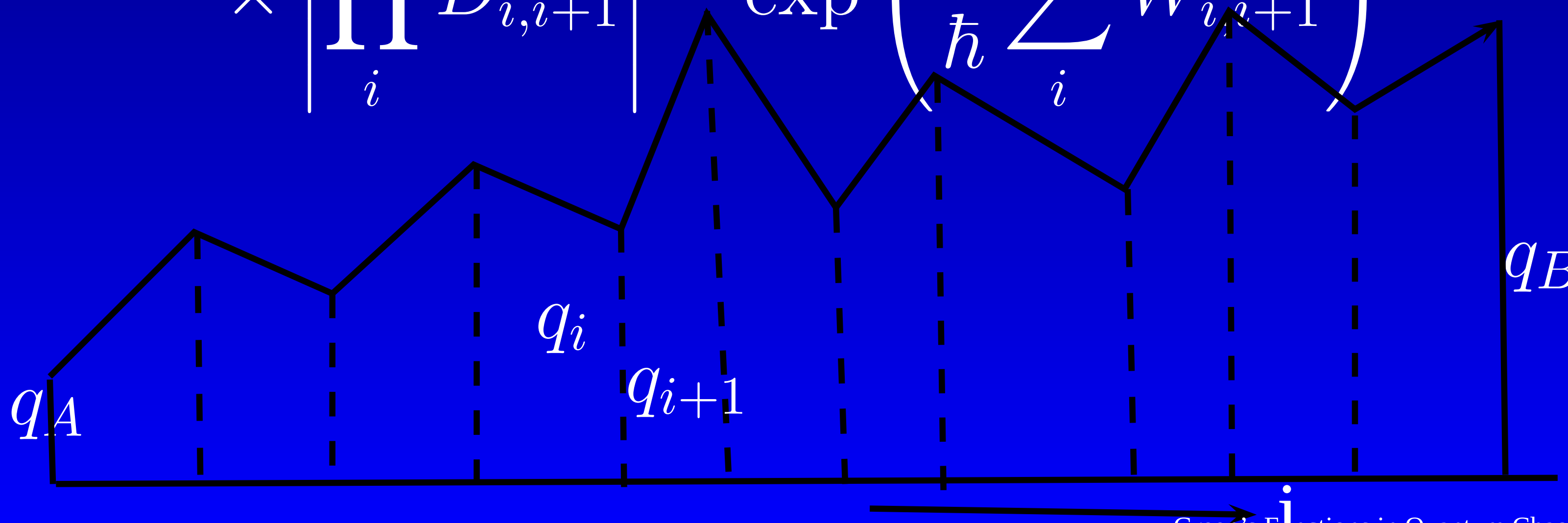

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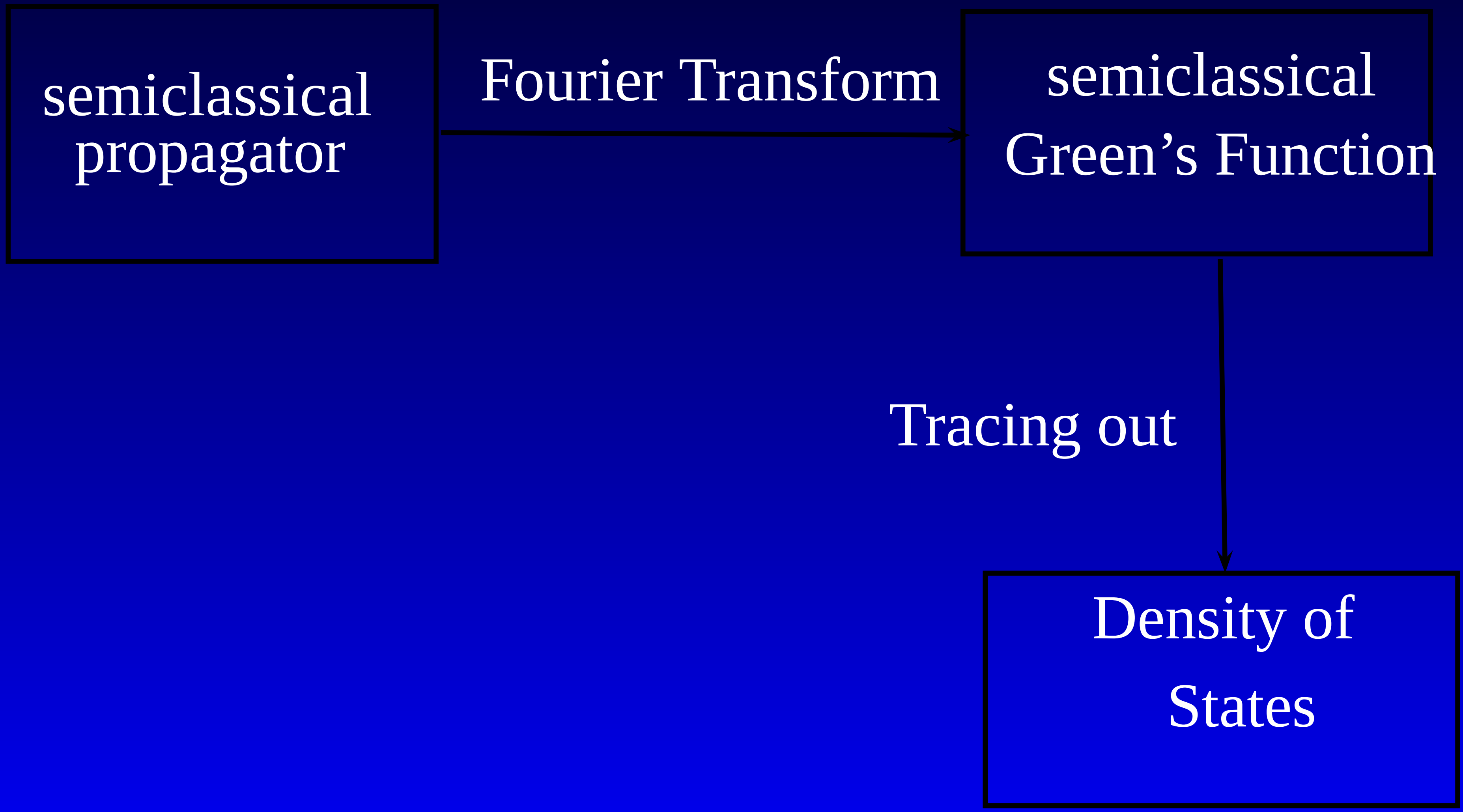
This integral can be simplified using semiclassical approximation.

Roadmap


semiclassical propagator
Fourier Transform
semiclassical Green's Function
Tracing out
Density of States

Semiclassical Propagator

Semiclassical approximation holds when $\hbar$ is much much smaller in comparison with $W_{i,i+1}$ so that the contribution to the integral comes only when $\delta(\sum_i W_{i,i+1}) = 0$.

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This approximation effectively means contribution comes from the path for which $\delta W_{AB} = \delta \int_0^t L dt = 0$ i.e. from the classically allowed paths.

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$$K(q_B, q_A, t) = \left(\frac{1}{2\pi\imath\hbar}\right) \sum_r \mid D_{BA,r} \mid^{1/2} \times \exp\left(\frac{\imath}{\hbar} W_{BA,r}(t) - \imath\frac{\nu_r \pi}{2}\right);$$

Semiclassical Green's Function

Now taking Fourier transform of this propagator, we get the semiclassical Green's function i.e.,

$$G_{SC}(q_A, q_B, E) = \frac{i}{\hbar} \int K(q_B, q_A, t) \exp(iEt/\hbar) dt,$$

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Semiclassical Green's Function

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$$G_{SC}(q_A, q_B, E) = -\frac{i}{\hbar}\left(\frac{1}{2\pi\hbar}\right)^{\frac{d-1}{2}} \sum_r \mid \Delta_{BA,r} \mid^{\frac{1}{2}}$$

$$\times \exp\left[\frac{i}{\hbar} S_r(q_A, q_B, E) - \imath \frac{\nu_r \pi}{2}\right];$$

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where

$$|\, \Delta_{BA,r} \,| = \frac{|\, D_{BA,r} \,|}{|\, \partial^2 W_{BA} / \partial t^2 \,|}$$

Density of States

Hence

$$\rho(E) = \Im\left[\frac{1}{\pi}\int G_{SC}(q,q,E)dq\right]$$

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Again using stationary phase approximation,which demands for contributing paths,

$$p_A\big|_q = p_B\big|_q,$$

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$$\begin{aligned}\rho(E) &= \frac{1}{\pi}\Im\left[\int G_{SC}(q,q,E)d\vec{q}\right] \\ &= \frac{1}{\pi\hbar}\sum_r \frac{(T_p)_r}{||M_r-1||^{1/2}}\cos\left[\frac{S_r(E)}{\hbar}-\frac{\mu_r\pi}{2}\right].\end{aligned}$$

This is called Gutzwiller Trace formula.

END

OK, it's over! Thank you!

References

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